Tenta Översiktskurs i Analys

090318

1a/. $\int 2x\cos x\,dx = 2\int x\cos x\,dx = \begin{bmatrix} g(x)=x & dg(x)=dx \\ df(x)=\cos(x) & f(x)=\sin(x) \end{bmatrix}$

$= 2x\sin x - 2\int \sin x\,dx = 2x\sin x + 2\cos x + C.$

b) $f(x) = \sqrt{x^2-3x} + x$

$f'(x) = \dfrac{2x-3}{2\sqrt{x^2-3x}} + 1.$

c) $\int_{-\infty}^{0}\left(\dfrac{2x-3}{2\sqrt{x^2-3x}} + 1\right)dx \underset{\uparrow}{=} \lim_{R\to-\infty}\left[\sqrt{x^2-3x}+x\right]_R^0 =$

enl. (a)

$= -\lim_{R\to\infty} \dfrac{\left(\sqrt{R^2-3R}+R\right)\left(\sqrt{R^2-3R}-R\right)}{\sqrt{R^2-3R}-R} =$

$= -\lim_{R\to\infty} \dfrac{R^2-3R-R^2}{|R|\sqrt{1-3/R}-R} = -\lim_{R\to-\infty} \dfrac{-3R}{-R\left(\sqrt{1-3/R}+1\right)} =$

$= -\lim_{R\to-\infty} \dfrac{3}{\sqrt{1-\underbrace{3/R}_{\to 0}}+1} = -\dfrac{3}{2}.$

2) $\begin{cases} x(t) = r\cos(t) \\ y(t) = r\sin(t) \end{cases} \Rightarrow \quad \begin{matrix} x'(t) = -r\sin(t) \\ y'(t) = r\cos(t) \end{matrix}$

Längden ges av

$$\int_0^{2\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{r^2}\,\underbrace{\sqrt{\sin^2(t) + \cos^2(t)}}_{=1}\, dt$$

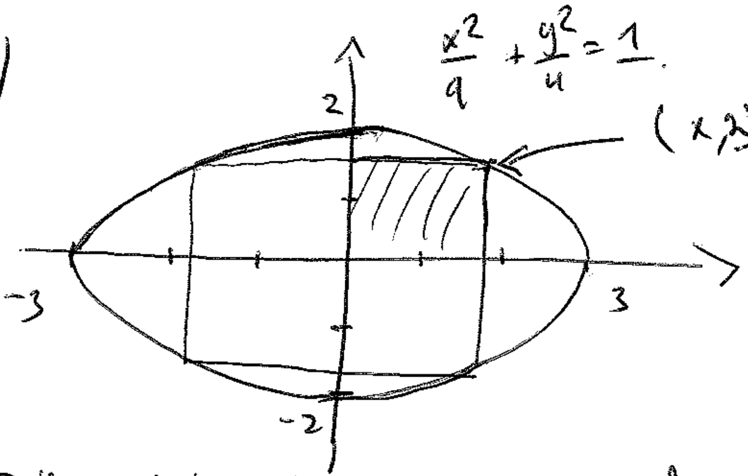
$$= \int_0^{2\pi} |r|\, dt \underset{r>0}{=} \int_0^{2\pi} r\, dt = r\big[dt\big]_0^{2\pi} = \underline{\underline{2\pi r}}$$

Kurvan är en cirkel med radie r.

3)

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\left(x, \frac{2\sqrt{9-x^2}}{3}\right)$



Tillräckligt att hitta arean för ▨ och multiplicera med 4.

Arean för ▨ ges av $x \cdot \frac{2}{3}\sqrt{9-x^2}$, så arean för hela rektangeln ges av

$$A(x) = 4 \cdot x \cdot \frac{2}{3}\sqrt{9-x^2} = \frac{8x\sqrt{9-x^2}}{3}$$

Derivering ger att

$$A'(x) = \frac{8}{3}\sqrt{9-x^2} + \frac{8x}{3} \cdot \frac{-\cancel{2}x}{\cancel{2}\cdot\sqrt{9-x^2}} =$$

$$= \frac{8}{3}\left(\sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}}\right).$$

Kritiska punkter får vi då $A'(x)=0$, dvs då

$$\sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}} = 0$$

$$\Leftrightarrow$$

$$9 - x^2 - x^2 = 0 \Leftrightarrow 2x^2 = 9 \Leftrightarrow x = \frac{3}{\sqrt{2}}$$

För $x = \frac{3}{\sqrt{2}}$ så får vi vår maximal area för A.

Arean är då $A\left(\frac{3}{\sqrt{2}}\right) = \frac{8\cdot\cancel{3}}{\cancel{3}\cdot\sqrt{2}}\sqrt{9-\frac{9}{2}} =$

$$= \frac{8\cdot\sqrt{18-9}}{\sqrt{2}\cdot\sqrt{2}} = \frac{24}{2} = \underline{\underline{12}}$$

4) $\begin{cases} y' = 3x^2(1+y^2) \\ y(0) = 0 \end{cases}$

$$\frac{dy}{dx} = 3x^2(1+y^2) \iff \frac{dy}{1+y^2} = 3x^2\,dx$$

Integrera:

$$\int \frac{dy}{1+y^2} = \int 3x^2 dx$$

$$\iff$$

$$\arctan(y) = x^3 + C.$$

$$\Rightarrow$$

$$y = \tan(x^3 + C)$$

$y(0) = 0 \Rightarrow \quad 0 = \tan(0+C) \iff \tan C = 0.$

dvs då $\sin C = 0$, så $C = \pi \cdot n \quad n \in \mathbb{Z}.$

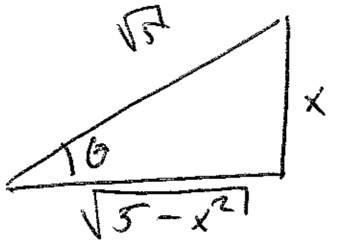
Detta ger att

$$\underline{\underline{y = \tan(x^3 + \pi \cdot n)}} \quad \text{för } n \in \mathbb{Z}.$$

5

5a/ $$\int_1^2 \frac{dx}{(5-x^2)^{3/2}} = \left[\begin{array}{l} x=\sqrt{5}\sin\theta \\ dx=\sqrt{5}\cos\theta\, d\theta \end{array}\right] =$$

$$= \int_{x=1}^{x=2} \frac{\sqrt{5}\cos\theta\, d\theta}{5^{3/2}\cos^3\theta} = \frac{1}{5}\int_{x=1}^{x=2} \frac{d\theta}{\cos^2\theta} = \frac{1}{5}\Big[\tan\theta\Big]_{x=1}^{x=2}$$



$$= \frac{1}{5}\left[\frac{x}{\sqrt{5-x^2}}\right]_1^2 = \frac{1}{5}\left(\frac{2}{\sqrt{1}} - \frac{1}{\sqrt{5-1}}\right) =$$

$$= \frac{1}{5}\left(2 - \frac{1}{2}\right)$$

$$= \underline{\underline{\frac{3}{10}}}$$

b). $$\int_1^{\ln 3} \frac{dx}{e^{2x}-4e^x+4} = \left[\begin{array}{l} u=e^x \\ du=e^x dx \Rightarrow \frac{du}{u}=dx \end{array}\right] =$$

$$= \int_e^3 \frac{du}{u(u-2)^2}$$

Partialbråksuppdelning:

$$\frac{1}{u(u-2)^2} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{u-2} = \frac{(A+B)u^2 + (-4A-2B+C)u + 4A}{u(u-2)^2}$$

$$\Rightarrow \begin{cases} A+B=0 \\ -4A-2B+C=0 \\ 4A=1 \end{cases} \Leftrightarrow A=1/4,\ B=-1/4,\ C=\frac{1}{2}.$$

$$\int_e^3 \frac{du}{u(u-2)^2} = \frac{1}{4}\int_e^3 \frac{du}{u} - \frac{1}{4}\int_e^3 \frac{du}{u-2} + \frac{1}{2}\int_e^3 \frac{du}{(u-2)^2} =$$

$$= \frac{1}{4}\Big[\ln u\Big]_e^3 - \frac{1}{4}\Big[\ln(u-2)\Big]_e^3 + \frac{1}{2}\Big[-\frac{1}{u-2}\Big]_e^3 =$$

$$= \frac{1}{4}\ln 3 - \frac{1}{4}\ln e - \frac{1}{4}\ln 1 + \frac{1}{4}\ln(e-2) - \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{e-2} =$$

$$= \frac{1}{4}\ln 3 - \frac{1}{4} - 0 + \frac{1}{4}\ln(e-2) - \frac{1}{2} + \frac{1}{2}\,\frac{1}{e-2} =$$

$$= \frac{1}{4}\ln 3 - \frac{3}{4} + \frac{1}{4}\ln(e-2) + \frac{1}{2}\cdot\frac{1}{e-2}.$$

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6)
$$\begin{cases} y'' + y' - 6y = e^{2x} \\ y'(0) = 1 \\ y(0) = 3 \end{cases}$$

Karakteristiska ekvationen ges av

$$0 = r^2 + r - 6 = (r+3)(r-2)$$

Detta ger att den homogena ~~ekvationen~~ lösningen är

$$y_H = C_1 e^{-3x} + C_2 e^{2x}.$$

Vi ansätter därför den partikulära lösningen y_p att vara

$$y_p = C x e^{2x}.$$

Insättning ger att

$$e^{2x} = y_p'' + y_p' - 6y_p = \left(Ce^{2x} + 2Cxe^{2x}\right)' + \left(Ce^{2x} + 2Cxe^{2x}\right)$$

$$-6Cxe^{2x} = 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} + Ce^{2x} + 2Cxe^{2x}$$

$$-6Cxe^{2x} = 5Ce^{2x}$$

$$\Longleftrightarrow$$

$$C = 1/5 . \Rightarrow y = C_1e^{-3x} + C_2e^{2x} + \frac{1}{5}xe^{2x}$$

Insättning av begynnelsevärden ger att:

$$1 = y'(0) = -3C_1e^{-3x} + 2C_2e^{2x} + \frac{1}{5}e^{2x} + \frac{2x}{5}e^{2x}\Big|_{x=0} =$$

$$= -3C_1 + 2C_2 + \frac{1}{5}$$

$$3 = y(0) = C_1 + C_2$$

Detta ger att
$$\begin{cases} -3C_1 + 2C_2 = \frac{4}{5} \\ C_1 + C_2 = 3 \Leftrightarrow C_1 = 3 - C_2. \end{cases}$$

$$\Rightarrow -3(3 - C_2) + 2C_2 = \frac{4}{5} \Leftrightarrow 5C_2 = \frac{4}{5} + 9$$

$$\Longleftrightarrow$$

$$C_2 = \frac{49}{25} \Rightarrow C_1 = \frac{75-49}{25} = \frac{26}{25}$$

Detta ger att lösningen är

$$y = \frac{26}{25}e^{-3x} + \frac{49}{25}e^{2x} + \frac{1}{5}xe^{2x}.$$

7a) $f(x) = \sqrt{x} \Rightarrow f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{4x^{3/2}} \Rightarrow f(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8x^{5/2}} \Rightarrow f'''(1) = \frac{3}{8}$.

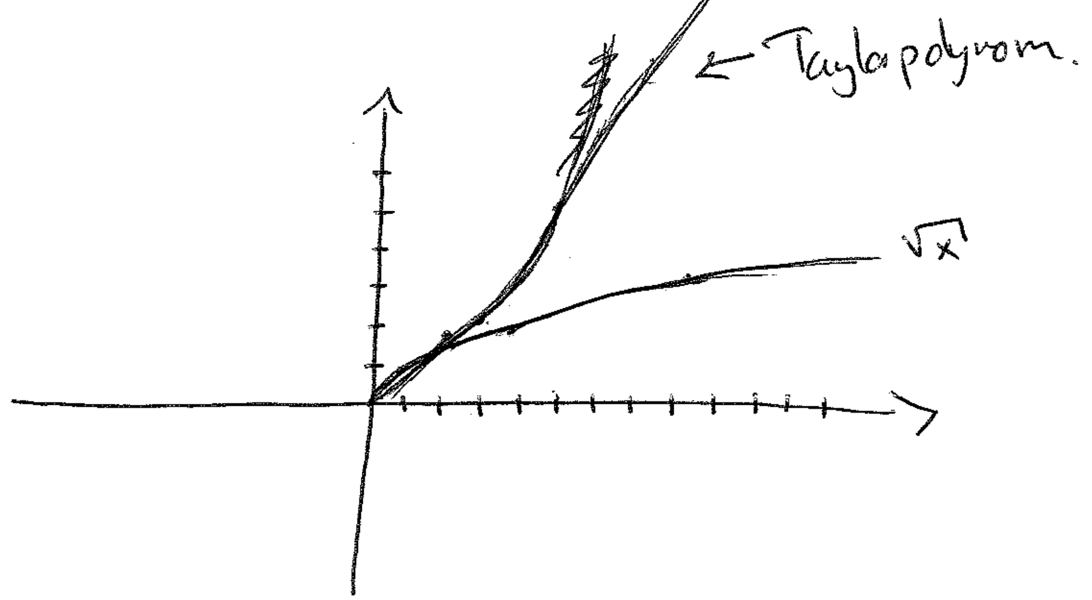
Detta ger att Taylorpolynomet ges av

$$f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{3!} +$$

$$+ R_4(x) =$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + R_4(x)$$

b).

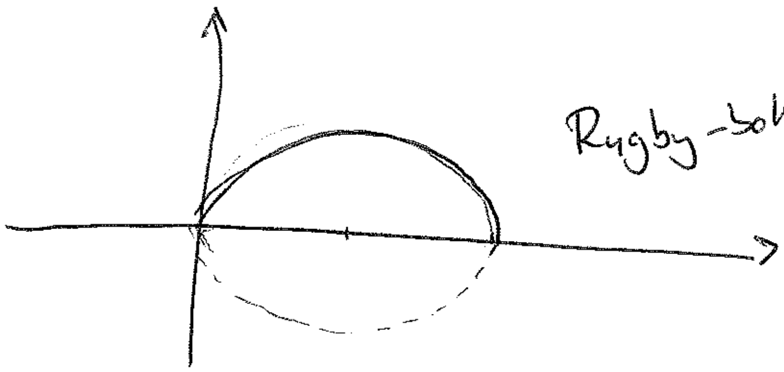


8). x-axeln:

$$\int_0^\pi \pi\cdot\sin^2(x)\,dx = \pi\int_0^\pi \left(\frac{1-\cos 2x}{2}\right)dx =$$

$$= \frac{\pi}{2}\int_0^\pi dx - \frac{\pi}{2}\int_0^\pi \cos 2x\,dx = \frac{\pi}{2}\Big[x\Big]_0^\pi - \frac{\pi}{4}\Big[\sin 2x\Big]_0^\pi =$$

$$= \frac{\pi^2}{2} + \cancel{\frac{\pi}{2}\cdot\frac{\pi}{2}} = \frac{\pi^2}{2}\cancel{+\pi} \quad \text{v.e.}$$



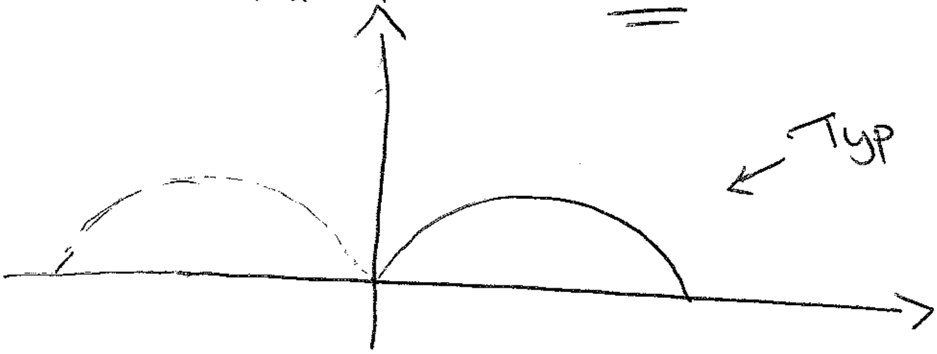
Rugby-boll.

y-axeln:

$$2\pi\int_0^\pi x\sin(x)\,dx = \begin{bmatrix} g(x)=x & dg(x)=dx \\ df(x)=\sin(x) & f(x)=-\cos(x) \end{bmatrix} =$$

$$= -2\pi\Big[x\cos x\Big]_0^\pi + 2\pi\int_0^\pi \cos x\,dx = -2\pi\Big[x\cos x\Big]_0^\pi + 2\pi\Big[\sin x\Big]_0^\pi =$$

$$= 2\pi\cdot\pi = \underline{\underline{2\pi^2}} \quad \text{v.e.}$$



Typ sockerkaka.