A direct approach to the study of soliton equations

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1 Introduction

As soliton theory is traditionally closely related to physical applications, explicit knowledge of particular solutions is much more emphasized than in many other branches of modern mathematics. It is astonishing how many different directions of mathematics, for example spectral theory, algebraic and differential geometry, are used to this purpose.

In the present article we shall propose a functional analytic approach to the construction of solutions, which has been elaborated at the university of Jena for the last six years. We shall try to place our work in the context of related actual research. Nevertheless, the following text can not be meant as a general introduction to functional analytic methods in soliton theory.

Next we sketch the organisation of the paper.

In Section 2 we explain the fundamental construction of our solution formulas taking the Korteweg-de Vries equation as prototypical example. This gives us the opportunity to examine the necessary tools from functional analysis.

Furthermore we discuss relations to the pioneering work of Marchenko and the bilinear method of Hirota. At the end of the section we give a schematic survey of results for other well-known soliton equations (among others the sine-Gordon equation, the nonlinear Schrödinger equation, and the Toda lattice).

In Section 3 we turn to two main applications. Inserting (finite-dimensional) matrices in our solution formula, we obtain the so-called negatons, a solution class, which was discovered and in special cases qualitatively described by Matveev. It is a merit of our approach to correlate the asymptotic properties of the solutions completely to the algebraic invariants of the matrix, confirming thereby expectations of Matveev. Thereafter we discuss solutions originating from diagonal operators. Here we obtain generalizations of deep results of Gesztesy et al. on countable nonlinear superpositions of solitary waves.

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Section 4 contains two supplementary aspects. Following joint work of the first author with Huang, we point out how to exploit C_0-semigroups of unbounded operators in order to refine the results on diagonal operators. In conclusion we report on the results of the thesis of Blohm where the relation to the inverse scattering transform is established.

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2 The underlying strategy

2.1 Discussion

The guiding principle of our approach to the study of soliton equations can be described as follows:

> *One translates a scalar equation and a special solution simultaneously to an operator equation and a corresponding operator-valued solution. Then one tries to regain scalar solutions by the use of an appropriate functional. Typically, these solutions now depend on an operator-valued parameter.*

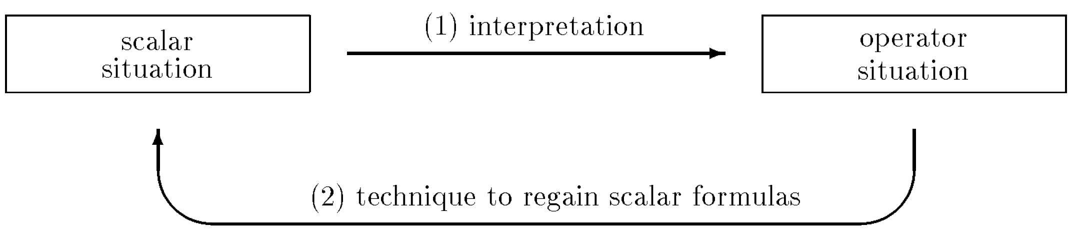
The original idea of the above strategy is due to Marchenko, who pursued it in his pioneering work [30] working with the function algebra

$$C^\infty(\mathcal{L}(H)) = \{T : \mathbb{R}^2 \to \mathcal{L}(H) \mid T \text{ infinitely differentiable}\},$$

$\mathcal{L}(H)$ the bounded linear operators on the Hilbert space H.

In contrast to his approach, we use the concept of operator ideals as introduced by Pietsch in [45]. In particular, this means that we may bring in the theory of Banach spaces instead of Hilbert spaces and that, in the applications, we have strong results about the factorization of operators at our disposal. For this it is crucial that we are not necessarily dealing with operators with common source and target space.

To carry out the strategy, as indicated in the following diagram, one has to overcome two different problems.



More precisely, we have to perform the subsequent steps:

(1) To formulate operator versions of equations

In fact, it is one of the main problems of our strategy to find a formulation of the investigated equation allowing an operator-valued interpretation. In general, one encounters situations, which do not allow a straightforward translation to the operator level.

(2) To provide techniques of regaining scalar solutions from operator solutions

A reasonable idea is to apply an appropriate functional τ to the operator solution deduced in the first step. To guarantee that the resulting scalar function is indeed a solution of the considered equation, the functional has to satisfy certain multiplicity properties.

This point causes difficulties. Already on the smallest operator ideal $\mathcal{F}$ of finite rank operators there does not exist a non-vanishing linear functional τ with the property

$$\tau(QR) = \tau(Q)\tau(R) \qquad \forall\, Q, R \in \mathcal{F}(E),$$

E a Banach space. We overcome this difficulty by assuming the operators relevant for the calculation to be one-dimensional. In this situation, the right choice for the functional τ is the trace as a discussion in Aden/Carl [3] shows.

Proposition 2.1 *([3], Proposition III A 2) Let E be a Banach space and $\tau : \mathcal{F}(E) \to \mathbb{C}$ a non-vanishing linear functional with the multiplicity property*

$$\tau\Big((a \otimes c)^2\Big) = \Big(\tau(a \otimes c)\Big)^2$$

for one-dimensional operators $a \otimes c$. Then, on the component $\mathcal{F}(E)$ of finite rank operators, τ is necessarily the trace tr.

2.2 Outline of arguments for the Korteweg-de Vries equation

Next, we implement the strategy discussed above exemplarily for the Korteweg-de Vries equation,

$$u_t = u_{xxx} + 6uu_x, \tag{1}$$

KdV for short.

In [3] this strategy was for the first time established for an integrated form of the KdV, for the presentation of the results we shall closely follow [13].

(1) Formulation of an operator-valued KdV

As a starting point, we have so far always chosen the 1-soliton, a comparatively simple solution. For the KdV, its operator-valued interpretation is given in the following proposition.

Proposition 2.2 *(cf. [3], Theorem III B 2 (i); [13], Proposition 5.1) Let E be a Banach space, $A \in \mathcal{L}(E)$. For $B \in \mathcal{A}(E)$, $\mathcal{A}$ a quasi-Banach operator ideal, we define $L(x,t) := \exp(Ax + A^3t)B$. Then*

$$U := \Big((1+L)^{-1}(AL + LA)\Big)_x \in \mathcal{A}(E)$$

is a solution of the operator KdV in $\mathcal{A}$,

$$U_t = U_{xxx} + 3(UU_x + U_xU), \tag{2}$$

provided that $(1+L)^{-1}$ exists.

For the KdV, the translation to the operator level is quite natural. Already for its discretization, the Toda lattice, the translation is much more involved. Some examples, for which a straightforward translation is not possible, are put together in Section 2.8.

Remark 2.3 *An approach of entirely different kind, as well basing essentially on functional analytic methods, was developed by Fuchssteiner. His first step is to read a nonlinear integrable system as an ordinary differential equation of the form*

$$u_t = K[u],$$

where u is a point of an infinite-dimensional manifold M of space-dependent functions and K a vector field on M. Then the solutions of the evolution equation correspond to those time-dependent curves $u(t)$ in M which satisfy the equation. On this level it is possible to apply methods from symplectic geometry to construct integrals of motion (conserved quantities), recursion operators, hierarchies, and integrable deformations of integrable systems (cf. for example [18], [19] for the KdV).

A quantized version of the KdV, which was stated by Fuchssteiner in a joint article [20] with Chowdhury, seems to be in close correlation with our results. The authors derive an equation, which formally coincides with our operator equation (2). By a formal calculus, they deduce a bihamiltonian form, which is then used to construct infinitely many conserved quantities of the Quantum KdV, thereby proving integrability in the classical sense.

Finally they consider a subtle algebra of distributions, which allows a complete realization of the above model.

Remark 2.4 *Nonabelian generalizations of soliton equations are the subject of numerous publications, but usually they appear in a completely different context: One considers matrix equations for matrices of a fixed dimension M, interpreted as a system of M equations, and tries to extend the inverse scattering transform to this system. The purpose of these generalizations of the inverse scattering transform is to integrate as large as possible classes of equations into the formalism. First results of this type are due to Wadati and Kamijo [57].*

(2) Deduction of a scalar solution formula

The following, rather general statement is fundamental to recover scalar formulas.

Proposition 2.5 *([3], Proposition III B 1; [13], Proposition 5.2) Let $\mathcal{A}$ be a quasi-Banach operator ideal with a continuous trace τ. If $U = U(x,t)$ is a solution of the operator KdV (2) in $\mathcal{A}$, which in addition satisfies $UP = U$, $U_x P = U_x$ for a projection P with $\operatorname{rank}(P) = 1$, then*

$$u = \tau(U)$$

is a solution of the scalar KdV (1).

Next we apply Proposition 2.5 to the operator solution of Proposition 2.2.

Moreover, we give a direct reformulation of the resulting solution formula, which is especially useful for explicit calculations since it completely avoids the involved evaluation of the inverse operator $(1 + L)^{-1}$.

Theorem 2.6 *(cf. [3], Theorem III B 2 (ii); [13], Theorem 5.4) Let $\mathcal{A}$ be a quasi-Banach operator ideal with a continuous determinant δ and E be a Banach space. Define again*

$$L(x,t) := \exp(Ax + A^3 t)B \text{ with } A \in \mathcal{L}(E) \text{ and } B \in \mathcal{A}(E),$$

where A and B are chosen such that $\operatorname{rank}(AB + BA) = 1$.

Then a solution of the KdV (1) is given by

$$u = \operatorname{tr}\Big(\big(\,(1+L)^{-1}(AL+LA)\,\big)_x\Big) = 2\frac{\partial^2}{\partial x^2}\log\delta(1+L) \tag{3}$$

provided that $\delta(1+L) \neq 0$.

2.3 Excursion about traces on operator ideals

The most important tool for the deduction of scalar solution formulas is the theory of traces and determinants on operator ideals, see the book of Pietsch [47], Chapter 4, for a good introduction. In addition, we want to refer to the monographs of Defant/Floret [15], König [29], Pietsch [45], and Simon [55], which are of fundamental importance in this context.

Let us start from the smallest operator ideal $\mathcal{F}$ of finite rank operators, on which the existence of a unique trace tr is well-known. There are two possibilities to express this trace,

a) $\operatorname{tr}(T) = \sum_{i=1}^n \langle y_i, a_i\rangle$, where $\sum_{i=1}^n a_i \otimes y_i$ is an arbitrary (finite) representation of T,

b) $\operatorname{tr}(T) = \sum_{i=1}^N \lambda_i(T)$, where $\lambda_i(T)$ denote the eigenvalues of T
(tr is a spectral trace),

and each of these possibilities immediately leads to an ansatz to extend the trace to (larger) quasi-Banach operator ideals.

The extension indicated in **b)** concerns operators $T \in \mathcal{L}(E)$ with absolutely summing eigenvalues, $\sum_i \mid \lambda_i(T) \mid < \infty$. Of course we always assume T to be a Riesz operator in order to guarantee the existence of the eigenvalue sequence $(\lambda_i(T))_i$.

This ansatz leads to quasi-Banach operator ideals $\mathcal{A} \subseteq \mathcal{S}_1^{eig}$ of so-called eigenvalue type 1, where $\mathcal{S}_1^{eig} = \bigcup_{E,F} \mathcal{S}_1^{eig}(E,F)$ denotes the class consisting of the sets

$$\mathcal{S}_1^{eig}(E,F) = \left\{ T \in \mathcal{L}(E,F) \;\middle|\; \begin{array}{c} ST \text{ is Riesz possessing absolutely summing} \\ \text{eigenvalues } \forall S \in \mathcal{L}(F,E) \end{array} \right\}.$$

Unfortunately, $\mathcal{S}_1^{eig}$ itself is not an operator ideal. This is due to the fact that a Banach space E can be found such that $\mathcal{S}_1^{eig}(E,E)$ is not a vector space. Nevertheless, if we restrict considerations to the infinite-dimensional separable Hilbert space H, then $\mathcal{S}_1^{eig}(H,H)$ becomes an operator ideal, which coincides with the well-known Schatten class of type ℓ_1 (confer Pietsch [46] for these statements).

The following deep result of White clarifies the existence of a spectral trace on ideals $\mathcal{A}$ of eigenvalue type 1.

Proposition 2.7 *([58]) Let $\mathcal{A}$ be a quasi-Banach operator ideal with $\mathcal{A} \subset \mathcal{S}_1^{eig}$. For arbitrary Banach spaces E and every operator $T \in \mathcal{A}(E)$ we define*

$$\operatorname{tr}_\lambda(T) := \sum_{i=1}^\infty \lambda_i(T).$$

Then the function $\operatorname{tr}_\lambda$ *is a continuous and spectral trace on* $\mathcal{A}$.

In general, the spectral trace tr_λ is not unique as an observation of Kalton shows, who proved in [28] the existence of a quasi-Banach operator ideal $\mathcal{A} \subset \mathcal{S}_1^{eig}$ admitting different continuous traces.

Finally, we want to remark that a maximal quasi-Banach operator ideal $\mathcal{A} \subset \mathcal{S}_1^{eig}$ (i.e. any quasi-Banach operator ideal in $\mathcal{S}_1^{eig}$ containing $\mathcal{A}$ already coincides with $\mathcal{A}$) does not exist (see Pietsch [47]).

The representation in **a)** motivates the extension of the trace to the class $\mathcal{N}_r = \cup_{E,F}\mathcal{N}_r(E,F)$ of so-called r-nuclear operators in the sense of Grothendieck ($0 < r \leq 1$). To recall the notion, an operator $T \in \mathcal{L}(E,F)$ belongs to $\mathcal{N}_r(E,F)$ if and only if

$$\exists\, a_i \in E', y_i \in F: \quad T = \sum_{i=1}^{\infty} a_i \otimes y_i \quad \text{with} \quad \sum_{i=1}^{\infty} \| a_i \|^r \cdot \| y_i \|^r < \infty.$$

$\mathcal{N}_r$ becomes an r-Banach operator ideal with respect to the r-norm

$$\| T \mid \mathcal{N}_r \| := \inf \Big\{ \Big(\sum_{i=1}^{\infty} \| a_i \|^r \cdot \| y_i \|^r \Big)^{\frac{1}{r}} \Big\} \qquad \text{for } T \in \mathcal{N}_r(E,F),$$

where the infimum is taken over all possible representations of T.

$\mathcal{N}_r$ is the smallest r-Banach operator ideal.

Proposition 2.8 *On the Banach operator ideal $\mathcal{N}$ restricted to the class of all Banach spaces with approximation property*

$$\mathrm{tr}_{\mathcal{N}}(T) := \sum_{i=1}^{\infty} \langle y_i, a_i \rangle \quad \textit{for } T = \sum_{i=1}^{\infty} a_i \otimes y_i \quad (a_i \in E', y_i \in E)$$

defines a unique continuous trace.

In this context, the decisive question was whether $\mathrm{tr}_{\mathcal{N}}(T)$ is independent of the representation of T. For arbitrary Banach spaces the answer is negative, a problem which had remained open for a long time. It was finally solved by Enflo in [16], who constructed a Banach space without approximation property.

Also in [16], Enflo showed the existence of an operator $S \in \mathcal{N}(\ell_1)$ with $\mathrm{tr}_{\mathcal{N}}(S) = 1$ and $S^2 = 0$. As a direct consequence, the trace $\mathrm{tr}_{\mathcal{N}}$ is not spectral.

On the other hand, from $\mathcal{N}_r \subset \mathcal{S}_1^{eig}$ for $0 < r \leq \frac{2}{3}$, it is clear that, on the smaller r-Banach operator ideals $\mathcal{N}_r$ ($0 < r \leq \frac{2}{3}$), there even exists a spectral trace tr_λ. Moreover, in this situation

$$\mathrm{tr}_{\mathcal{N}_r}(T) := \sum_{i=1}^{\infty} \langle y_i, a_i \rangle \quad \text{for } T = \sum_{i=1}^{\infty} a_i \otimes y_i \in \mathcal{N}_r(E)$$

yields a trace on $\mathcal{N}_r$, which is defined with respect to the class of all Banach spaces without any restriction. This trace is unique and thus coincides with the spectral trace.

The relationship between traces and determinants is governed by the 'Trace-determinant theorem' (see Pietsch [47], Grobler et al. [23]).

Proposition 2.9 *There is a one-to-one correspondence between continuous traces and continuous determinants on every quasi-Banach operator ideal.*

Proposition 2.10 *Let δ be a continuous determinant on a quasi-Banach operator ideal $\mathcal{A}$. If the $\mathcal{A}(E)$-valued function $T(z)$ is differentiable at a point z_0 with respect to the quasi-norm $\| \cdot \mid \mathcal{A} \|$, then $\delta(I + T(z))$ is differentiable at z_0 as well, and the formula*

$$\frac{\partial}{\partial z}\Big(\delta(I + T(z_0))\Big) = \tau\Big((I + T(z_0))^{-1}\frac{\partial}{\partial z}T(z_0)\Big)\ \delta\Big(I + T(z_0)\Big)$$

holds whenever $I + T(z_0)$ is invertible.

Here τ is the corresponding trace given by $\tau(S) := \lim_{z\to 0}\frac{1}{z}\Big(\delta(I + zS) - 1\Big)$ for $S \in \mathcal{A}(E)$.

A detailed introduction to the above material can be found in [13].

2.4 The operator equation $A_2X + XA_1$

Next we turn to the rank condition of Theorem 2.6. To include also the soliton equations treated in Section 2.8, we formulate it in a slightly more general form:

> *Given $A_j \in \mathcal{L}(E_j)$, E_j Banach spaces, for $j = 1, 2$. Which assumptions on A_j are sufficient to ensure the existence of an operator X satisfying $\operatorname{rank}(A_2X + XA_1) = 1$? What can be said about the properties of X?*

To this end, we consider the operator $\Phi_{A_1,A_2} : \mathcal{A}(E_1, E_2) \longrightarrow \mathcal{A}(E_1, E_2)$, $\mathcal{A}$ a quasi-Banach operator ideal, defined by

$$\Phi_{A_1,A_2}(X) := A_2X + XA_1 \tag{4}$$

and ask for its spectrum, $\operatorname{spec}(\Phi_{A_1,A_2}) := \{\lambda \in \mathbb{C} | \,(\lambda I - \Phi_{A_1,A_2}) \text{ is not invertible in } \mathcal{L}(\mathcal{A}(E_1, E_2))\}$.

In fundamental papers Eschmeier [17] and Dash/Schechter [14] determined the spectrum of an elementary operator Φ,

$$\Phi = p(R_{A_1}, L_{A_2}), \tag{5}$$

where p is a polynomial and R_{A_1}, L_{A_2} denote the multiplication with $A_1 \in \mathcal{L}(E_1)$ from the right-hand and $A_2 \in \mathcal{L}(E_2)$ from the left-hand side, respectively. Here we cite a result of Aden extending a part of their result from Banach operator ideals to p-Banach operator ideals $(0 < p \leq 1)$.

Proposition 2.11 *([1]; [2], Theorem III.2.7) Let $\mathcal{A}$ be a p-Banach operator ideal $(0 < p \leq 1)$. Then the spectrum of the operator Φ defined in (5) is given by*

$$\operatorname{spec}(\Phi) = p(\operatorname{spec}(A_1), \operatorname{spec}(A_2)) \tag{6}$$

independently of the underlying p-Banach operator ideal $\mathcal{A}$.

Corollary 2.12 *Suppose $0 \notin \operatorname{spec}(A_1) + \operatorname{spec}(A_2)$. Then, for every operator $C \in \mathcal{A}(E_1, E_2)$, the equation $A_2X + XA_1 = C$ in $\mathcal{A}(E_1, E_2)$, considered in any arbitrary quasi-Banach operator ideal $\mathcal{A}$, has a unique solution X, namely*

$$X := \Phi^{-1}_{A_1,A_2}(C) \in \mathcal{A}(E_1, E_2).$$

For a systematic approach to solve the operator equation $A_2X + XA_1$ without knowledge of $\mathrm{spec}(A_j)$ $(j = 1, 2)$, the subsequent lemma is useful.

Lemma 2.13 *Let $\mathcal{A}$ be a Banach operator ideal and $A_j \in \mathcal{L}(E_j)$, E_j Banach spaces $(j = 1, 2)$. If*

$$\int_0^\infty \| e^{-A_2 s} C e^{-A_1 s} \mid \mathcal{A} \| \, ds < \infty \quad \text{for} \quad C \in \mathcal{A}(E_1, E_2),$$

then the operator X defined by the indefinite integral $X := \int_0^\infty e^{-A_2 s} C e^{-A_1 s} \, ds \in \mathcal{A}(E_1, E_2)$ is a solution of the equation $A_2X + XA_1 = C$.

Remark 2.14 *The above lemma can not be transferred to quasi-Banach operator ideals $\mathcal{A}$ as a counterexample by T. Kühn (explained in [2], Section IV.2) shows.*

The following criterion is fundamental for the extension of our method to unbounded operators in [12], see also Section 4.1.

Proposition 2.15 *Let A_j be generators of C_0-semigroups $\left(T_j(x)\right)_{x \geq 0}$ on Banach spaces E_j, $j = 1, 2$, with $0 \notin \overline{\mathrm{spec}(A_1) + \mathrm{spec}(A_2)}$. Then*

$$\mathcal{T}_{1,2}(x)X = \Big(T_2(x) X T_1(x) \Big)_{x \geq 0} \qquad \text{for } X \in \mathcal{K}(E_1, E_2)$$

defines a C_0-semigroup on the compact operators $\mathcal{K}(E_1, E_2)$, and its generator Φ_{A_1,A_2} is formally given by $\Phi_{A_1,A_2}X = A_2X + XA_1$ (compare Section 4.1 as well). For the invertibility of Φ_{A_1,A_2}, each of the following conditions is sufficient:

(i) ([44], Theorem 3) $\sup_{x \geq 0} \| \int_0^x \mathcal{T}_{1,2}(y) dy \| < \infty$.

(ii) ([4], Theorem 4.1) A_j is the generator of a 'finally norm-continuous semigroup', i.e. there exists $x_0 > 0$ such that $\lim_{x \to x_0} \|T_j(x) - T_j(x_0)\| = 0$ $(j = 1, 2)$.

(iii) ([27]) $A_j = A$ for $j = 1, 2$ and the semigroup $\left(T_j(x)\right)_{x \geq 0} = \left(T(x)\right)_{x \geq 0}$ can be extended to a C_0-group $\left(T(x)\right)_{x \in \mathbb{R}}$ fulfilling the quasi-analytic growth condition

$$\int_{-\infty}^{\infty} \frac{\log(1 + \|T(x)\|)}{1 + x^2} dx < \infty.$$

The restriction to compact operators $\mathcal{K}(E_1, E_2)$ may as well be replaced with the restriction to nuclear operators $\mathcal{N}(E_1, E_2)$.

For a detailed discussion of the operator equation $A_2X + XA_1$, we refer also to the survey article of Bhatia and Rosenthal [7].

2.5 Marchenko's method

Next we shall explain Marchenko's method as introduced in his pioneering book [30]. His work was starting point and motivation to our investigations.

Roughly speaking, he treats nonlinear equations on a rather abstract level first, and then projects to the situation of actual interest. More precisely, Marchenko formulates this for the KdV as a prototypical example in the following way:

General strategy

Let $\Gamma(x,t)$ be an operator function defined on a domain of the (x,t)-plane, which is invertible on the whole domain and satisfies the conditions

$$\Gamma_t - 4\Gamma_{xxx} = 0, \qquad \Gamma_{xx} = a^2\Gamma, \tag{7}$$

$$\Gamma_x(1-P) \;=\; \Gamma N_0(1-P), \tag{8}$$

where a and N_0 are constant operators, and P denotes a one-dimensional projection ($P^2 = P$). Then

a) $U := 2(\Gamma^{-1}\Gamma_x)_x$ is an operator solution of the equation $U_t = U_{xxx} + 3(UU_x + U_xU)$,

b) the function u which is given by $PUP = uP$ is a scalar solution of the KdV (1).

Remark 2.16 *To deduce the operator equation, only (7) is used, whereas (8) is necessary to preserve the solution property of U under projection.*

Implementation

First of all, one chooses an arbitrary algebra of operator-valued functions, and in this algebra one looks for Γ with (7), (8). Obviously $\Gamma = e^{ax+4a^3t} + e^{-ax-4a^3t}M$ always satisfies (7), and, for $N_0 = a$, also (8), provided that $(aM + Ma)(1-P) = 0$. To fulfill the latter condition, it is enough to find operators M_0, R such that $aM_0 + M_0a = P$ and $[R,a] = 0$ holds, and to put $M = RM_0$. Therefore, principal technical difficulties in the implementation are a) to find a solution M_0 of the equation $aM_0 + M_0a = P$ for given a, P, and b) to guarantee that Γ is invertible.

Remark 2.17 *Our device is also motivated by Marchenko's strategy, but there are essential differences: The argument of Marchenko is built on the notion of a logarithmic derivative $\Gamma^{-1}\Gamma_x$. To project onto the solution, $U = UP + U(1-P)$ must decompose in a one-dimensional and a constant (actually $U(1-P) = N_0$) part. In contrast, we start from an explicit solution, namely the simplest possible, the 1-soliton, which we interpret as operator function. The non-commutativity of operators requires an additional symmetrization, which turns out to be also the right condition for projecting back.*

The operator N_0, which Marchenko can treat as an additional parameter by his slightly more general approach, must be arranged conveniently for each concrete realization. Anyway, Marchenko himself observes that this parameter is not very significant in his applications since it either (realization **a)***) does not contain any information at all or (realization* **b)***) coincides with the operator a.*

On the other hand, our approach leads to considerably more transparent solution formulas as we shall see below in the case of the N-solitons.

First of all, Marchenko separates a purely algebraic treatment of his method.

In an associative ring K with unity 1, Marchenko interpretes equations (7) in terms of generalized derivations ($\partial \in \mathcal{L}(K)$ is called generalized derivation, if it satisfies the product rule $\partial(k_1k_2) = (\partial k_1)k_2 + k_1(\partial k_2)$ for k_1, $k_2 \in K$) and deduces from that an analogue of the KdV in K. Under the assumption (8), every idempotent element $P \in K$ yields an operation of projection of the ring K onto its subring $K_0 = PKP$.

For the KdV, the initial ring K_0, in which solutions are sought, is $K_0 = C^\infty$.

Two particular realizations have been considered by Marchenko. In the sequel, we report on his results in this context.

a) Realization in matrix rings

The simplest extension of K_0 is the ring $K = \mathrm{Mat}_N(K_0)$ of $N \times N$-matrices with elements in K_0, and the initial ring K_0 is canonically identified with the subring PKP, where

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Extensions of generalized derivations ∂ of the ring K_0 to the ring $K = \mathrm{Mat}_N(K_0)$ are defined by $\partial k = (\partial k_{ij})_{i,j=1}^N$ for $k = (k_{ij})_{i,j=1}^N$.

The projection condition (8) can always be solved with so-called Wronsky matrices $W = W(\partial; f_1, \dots, f_N) := \left(\partial^{N-j} f_i\right)_{i,j=1}^N$. In fact $\partial W(1_N - P) = W N_0 (1_N - P)$, where

$$N_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdotsfor{5} \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and, moreover, $P(W^{-1}\partial W)P = wP$ with $w = (\det W)^{-1}\partial(\det W)$.

To get solutions from this, a matrix a has to be chosen. Then, solving the equations (7) for W, the functions f_j $(j = 1, \dots, N)$ involved in W can be determined. Finally, the invertibility of W has to be checked.

Starting from the diagonal matrix

$$a = \begin{pmatrix} \sqrt{k_1} & 0 & \cdots & 0 \\ 0 & \sqrt{k_2} & \cdots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \cdots & \sqrt{k_N} \end{pmatrix} \qquad \text{for } 0 < k_1 < \dots < k_N,$$

one gets the following formula for the N-solitons

$$u = 2\frac{\partial^2}{\partial x^2} \log \Big(\det W(\partial_x; f_1, \dots, f_N) \Big) \qquad \text{with } f_j = \frac{1}{2}(e^{\vartheta_j} + (-1)^{j-1} e^{-\vartheta_j})$$
$$\text{and } \vartheta_j(x,t) = k_j x + 4k_j^3 t + \varphi_j.$$

This representation of N-solitons by Wronsky determinants arises as well very naturally if solutions are constructed successively with the so-called Darboux transformation. For an introduction to the latter, we refer to the book of Matveev/Salle [33].

Remark 2.18 *Within our device, N-solitons are recovered by inserting the following $N \times N$-matrix in diagonal form as operator A:*

$$A = \begin{pmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \cdots & k_N \end{pmatrix}.$$

Marchenko also constructs solutions, which are no longer regular, starting from matrices a in Jordan canonical form. For a description of those solutions within our operator method, we refer to Subsection 3.1.

Remark 2.19 *In our approach, we achieve an additional flexibility, which is due to the fact that we use the concept of operator ideals. For one first time, this proves to be an advantage since we do not run into difficulities extending the solution class of N-solitons to superpositions of a countable number of solitons. Within the device of Marchenko, results of this type are not visible without further ado.*

We refer to the discussion in Subsection 3.2, where the significant role of the factorization results, which are available for the theory of Banach spaces, becomes apparent.

b) Realization in operator algebras

As initial ring K_0, Marchenko considers the algebra $C^\infty(\mathcal{L}(H_0))$ of infinitely differentiable functions of a domain in the (x,t)-plane into the bounded linear operators in H_0, H_0 a separable Hilbert space.

As extension K, he takes the algebra $C^\infty(\mathcal{L}(H))$, where $H = L^2_\mu(\Omega, H_0) = \{f : \Omega \to H_0 \mid ||f(z)||_{H_0} \in L^2_\mu(\Omega)\}$, (Ω,μ) a measurable space with $0 < \mu < \infty$.

Appropriate projections P with $PH = H_0$, thus $PC^\infty(\mathcal{L}(H))P = C^\infty(\mathcal{L}(PH)) = C^\infty(\mathcal{L}(H_0))$, arise from $P(f(z)) = \int_\Omega p(z)(f(z))d\mu(z)$ for $p \in L^2_\mu(\Omega, \mathcal{L}(H_0))$ with $\int_\Omega p(z)d\mu(z) = 1$.

Here, the integral $\int_\Omega f d\mu$ of $f \in L^2_\mu(\Omega, H_0)$ is defined as the element $h_0 \in H_0$ which generates the functional $(\cdot, h_0) = \int_\Omega (\cdot, f(z))_{H_0} d\mu(z)$ and the integral $\int_\Omega F d\mu$ of $F \in L^2_\mu(\Omega, \mathcal{L}(H_0))$ is the operator on H_0 which is given by $\left(\int_\Omega F d\mu\right)(h_0) := \int_\Omega (F(z)h_0)d\mu(z)$.

For the KdV, $\dim H_0 = 1$, that is $H_0 \simeq \mathbb{C}$, $C^\infty(\mathcal{L}(H_0)) \simeq \mathbb{C}$, and the above concept coincides with the usual conventions in $L^2_\mu(\Omega)$. Nevertheless, the implementation of the general strategy is extremly involved. Therefore, we only state the main result.

Proposition 2.20 *([30], Theorem 3.6.1) Let*

1) the support Ω of the measure μ lie in the union of the real and imaginary axis, with its asymmetric part $\Omega_1 = \{z | z \in \Omega, -z \notin \Omega\}$ being contained in a finite number of intervals $\Delta_k = [\,\mathrm{i}a_k^-, \mathrm{i}a_k^+\,]$ $(a_k^- < a_k^+)$ on the imaginary axis and satisfying $\operatorname{dist}(\Omega_1^+, \pi_1(\Omega_1^-)) > 0$ $(\pi_1(z) = -z)$*, where $\Omega_1^+ = \Omega_1 \cap \mathbb{C}^+$, $\Omega_1^- = \Omega_1 \backslash \Omega_1^+$;*

2) the nonnegative function $\omega(z)$ be a Muckenhoupt weight on the union γ of the real and imaginary axis, that is

$$\sup_{z' \in \gamma} \sup_{r>0} \left\{ \frac{1}{r} \int_{z \in B(z',r) \cap \gamma} \omega(z)|dz| \cdot \frac{1}{r} \int_{z \in B(z',r) \cap \gamma} \omega(z)^{-1}|dz| \right\} < \infty,$$

$B(z',r) := \{z : |z - z'| < r\}$, and satisfy the inequality $\inf_{z \in \Omega\backslash\Omega_1} (\omega(z)\omega(\overline{z})) > 0$*;*

3) the measure μ coincide with the linear Lebesgue measure $(d\mu(z) = (2\pi)^{-1}|dz|)$ on the set $\Omega\backslash\Omega_1$, while on the set Ω_1 it has to satisfy simultaneously both the $\omega(-z)$ and the $\omega(-z)^{-1}$ Carleson conditions, that is

$$\begin{aligned}
\mu([-\mathrm{i}h, \mathrm{i}h]) &\leq C \min\left\{ \int_{-h}^{h} \omega(-x)dx, \int_{-h}^{h} \omega(-x)^{-1}dx \right\}, \\
\mu([\mathrm{i}(a_k^\pm - h), \mathrm{i}(a_k^\pm + h)]) &\leq C \min\left\{ \int_{\mathrm{i}(a_k^\pm - h)}^{\mathrm{i}(a_k^\pm + h)} \omega(-z)|dz|, \int_{\mathrm{i}(a_k^\pm - h)}^{\mathrm{i}(a_k^\pm + h)} \omega(-z)^{-1}|dz| \right\};
\end{aligned}$$

4) the operator functions $p_0(z)$, $r(z)$, $r(z)^{-1}$, and $m(z)$ be bounded with $m^*(\overline{z}) = m(z)$, moreover $\sup_{-\infty<z<\infty} ||r^*(-\overline{z})m(z)r(-z)||_{H_0} + \frac{1}{2} \sup_{-\infty<z<\infty} \omega(z)^{-1}||p_0(z)r(z)||^2_{H_0} < 1.$

Then the operator $T = 1 + R(z)L$, *where*

$$R(z) = \Big(\mathbf{1}_{\Omega\setminus(\Omega\cap\mathbb{C}_+)}(z) - \mathbf{1}_{\Omega\cap\mathbb{C}_+}(z)\Big) r(z) r^*(-\overline{z})\, e^{-2\mathrm{i}z(x+4z^2 t)},$$

$$L(f(z)) = \Big(\mathbf{1}_{\Omega\setminus\Omega_1}(z)\Big) m(z) f(-z) + \int \frac{p^*(-\overline{z})p(z')}{\mathrm{i}(z+z')} f(z') d\mu(z'),$$

and $p(z) = \Big(\mathbf{1}_{\Omega_1}(z) + \mathbf{1}_{\Omega\setminus\Omega_1}(z)\omega(z)^{-\frac{1}{2}}\Big) p_0(z)$, $\mathbf{1}_{\widehat{\Omega}}(z)$ *the characteristic function on* $\widehat{\Omega}$, *is bounded and invertible, while*

$$U(x,t) = 2\frac{\partial^2}{\partial x^2} P\Big(- T^{-1} R(z) p^*(-\overline{z})\Big) P \quad \in C^\infty(\mathcal{L}(H_0))$$

is a self-adjoint solution of the KdV $\quad U_t = U_{xxx} + 3(UU_x + U_x U)$.

In the considered situation, $H_0 \simeq \mathbb{C}$, the operators $F \in L^2_\mu(\Omega, \mathcal{L}(H_0)) = L^2_\mu(\Omega, \mathcal{L}(\mathbb{C}))$ are identified with functions $f \in L^2_\mu(\Omega)$ according to $F(z)\lambda = \lambda(F(z)1) = \lambda f(z)$ for $\lambda \in \mathbb{C}$. Thus the solution of

$$\Big(- T^{-1} R(z) p^*(-\overline{z})\Big) P = g(z) P \iff \Big(- R(z) p^*(-\overline{z})\Big) P = \Big(T g(z)\Big) P$$

reduces to an integral equation for the function $g(z)$.

Two particular cases are considered:

(1) If $\Omega \cap \mathrm{i}\mathbb{R}$ consists of a finite number of points lying on the positive ray of the imaginary axis, then the integral equation for $g(z)$ coincides with the Gelfand-Marchenko equation of the inverse scattering transform.

(2) If $\Omega \cap \mathbb{R} = \emptyset$ and $\Omega \cap \mathrm{i}\mathbb{R}$ consists of a finite number of disjoint intervals lying symmetrically on the imaginary axis with respect to the origin, then the integral equation for $g(z)$ can be used to construct the Baker-Akhiezer function and to recover the algebro-geometric solutions.

Remark 2.21 *The attempt to integrate also other solution classes, which are well understood, in our formalism is very promising. Indeed, first results in this direction are already available, see Subsection 4.2.*

On the one hand, it should be possible to transfer the conditions on the support Ω *of the measure* μ *in natural manner to properties of the spectrum of the operator* A *which is central in our device and, thus, to contribute to the understanding of the geometrical properties of solutions (as for the negatons in Subsetion 3.1) or to study reasonable extensions of solution classes (see Subsection 3.2).*

On the other hand, it would be worthwhile to reduce the technical effort of Marchenko's method.

2.6 Bilinearization of Hirota

In order to explain the connection to Hirota's method, we roughly sketch his device for the KdV (see [24]). A more detailed presentation of the method and its extension to other soliton equations can be found in [25], [26].

The bilinear operator $D_x^i D_t^j$, $i, j \in \mathbb{N}_0$, introduced by Hirota is the mapping of the pair (g, h) to the function

$$D_x^i D_t^j\, g \cdot h := \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^i \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^j g(x,t)\, h(x',t') \Bigg|_{\substack{x=x'\\ t=t'}} .$$

For a discussion of its properties we also refer to the literature cited above.

Starting from a nonlinear equation in x, t, roughly speaking, Hirota's main idea is to set up a linear equation in twice as much variables x, x', t, t', and then, restricting certain solutions of the latter to the diagonal $x' = x$, $t' = t$, to obtain solutions of the considered equation again.

To this end one looks for an appropriate transformation of variables to rewrite the equation in terms of a bilinear operator. This is called bilinearization.

For the KdV, the bilinear form is

$$D_x(D_t - D_x^3)g \cdot g = 0, \tag{9}$$

and whenever g solves (9), a solution u of the KdV is obtained by $u = 2\frac{\partial^2}{\partial x^2} \log g$.

Remark 2.22 *Similar to our operator method, there is no algorithm to bilinearize an equation. In fact, it rather depends on the underlying equation which transformation of variables is suitable. Quite often, this transformation is motivated by a particular representation of the 1-soliton solution.*

To solve the bilinear equation, one uses a formal perturbation expansion. Inserting the expression

$$g(x,t) = \sum_{n=0}^{\infty} \epsilon^n g_n(x,t) \qquad \text{with } g_0(x,t) \equiv 1$$

and comparing coefficients in the powers of ϵ, the bilinear equation is turned into a hierarchy of equations for $g_n(x,t)$ $(n \geq 1)$, which can be solved successively.

In particular, starting from $g_1(x,t) = \sum_{j=1}^{N} \exp(\eta_j)$, the above series terminates. This yields the N-soliton solutions:

Proposition 2.23 *([24]) Let $0 < k_1 < \ldots < k_N$. The N-soliton solution of the bilinear KdV is given by*

$$g(x,t) = \sum_{\mu=0,1} \exp\Big(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{i<j}^{(N)} \mu_i \mu_j A_{ij}\Big),$$

$$\text{with } \eta_j = k_j x + k_j^3 t + \varphi_j \text{ and } e^{A_{ij}} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2,$$

where $\sum_{\mu=0,1}$ denotes summation over all possible vectors $\mu = (\mu_1, \ldots, \mu_N) \in \{0,1\}^N$.

The relation of Hirota's method to other well-known techniques in soliton theory, for example the inverse scattering transform or Bäcklund transformations, has been frequently studied. A comprehensive survey of results in this direction can be found in the book of Matsuno [32], where also some other types of solutions, which can be constructed with Hirota's method, are illustrated.

The following proposition shows that our method also yields solutions of Hirota's bilinear form.

Proposition 2.24 *([52], Chapter 2) Let $\mathcal{A}$ be a quasi-Banach ideal with continuous determinant δ.*

If $L(x,t) \in \mathcal{A}(E)$, E a Banach space, is a family of operators, which satisfies the base equations $L_x = AL$ and $L_t = A^3L$, and, in addition, $\operatorname{rank}(AL + LA) = 1$ holds, then

$$g(x,t) = \delta(I + L(x,t))$$

is a solution of the bilinear KdV (9).

2.7 An approach with Fredholm integral operators

Another approach, which is quite closely related to our method, has been developed by Pöppe. He used Fredholm determinants of certain integral operators to solve Hirota's bilinear form.

To provide Fredholm determinant theory (confer Smithies [56]) dealing with the semi-infinite interval $]-\infty,0]$ as well, Pöppe introduces the space

$$C_\nu := \Big\{\phi \in C^0(]-\infty,0],\mathbb{C}) \;\Big|\; \|\phi\|_\nu := \sup_{s\le 0}|\phi(s)|(1-s)^\nu < \infty\Big\}$$

for $\frac{1}{2} < \nu \le 1$. On C_ν, he defines a normed space LC_ν of Fredholm integral operators by

$$\begin{aligned} LC_\nu \;\; := \;\; & \Big\{K : C_\nu \to C_\nu \Big| K\phi(s) := \int_{-\infty}^0 k(s,\sigma)\phi(\sigma)d\sigma,\ k \text{ continuous}, \\ & \qquad \|K\| := M_\nu \sup_{s,\sigma\le 0}(1-s)^\nu(1-\sigma)^\nu|k(s,\sigma)| < \infty\Big\}, \end{aligned}$$

where $M_\nu := \int_{-\infty}^0 (1-s)^{-2\nu}ds < \infty$. Then, for every operator $K \in LC_\nu$, the Fredholm determinant $\det(I+\lambda K)$ is well-defined.

Proposition 2.25 *([49], Theorem 2.3) Let f be a solution of the linearized KdV $f_t = 8f_{xxx}$ decaying sufficiently fast as $x \to -\infty$ so that f and its derivatives up to order 4 with respect to x and order 2 with respect to t are in $C_{2\nu}$. This implies that the Fredholm integral operator $F(x,t)$ defined by*

$$F(x,t)\phi(s) := \int_{-\infty}^0 f(s+\sigma+2x,t)\phi(\sigma)d\sigma$$

as well as F_t, F_x, F_{xx}, F_{xxx} are in LC_ν for all x and t.

Then

$$g(x,t;\lambda) := \det\left(I+\lambda F(x,t)\right)$$

is a solution of the bilinear KdV (9) for every $\lambda \in \mathbb{C}$.

Remark 2.26 *N-soliton solutions are generated by the base function*

$$f(x,t) = \frac{\partial}{\partial x}\sum_{j=1}^N \exp\Big(\frac{k_j}{2}x + k_j^3 t + \varphi_j\Big),$$

an ansatz, which is similar to that of Hirota.

Pöppe has investigated the relationship of his approach to Bäcklund transformations and the inverse scattering transform. The reader may read up on those topics in [49], and, for further aspects, also in [50].

Remark 2.27 *a) The approach of Pöppe is based on the special choice of additive integral operators of a suitable arranged space. Using operator ideals, we avoid such a priori restrictions on the type of operators.*

b) In order to treat lattice equations, a different situation has to be considered. For the Toda lattice, for instance, the assumption in [5] on the involved operator $F_n(t) \in \mathcal{N}(E)$ (E a Banach space) is

$$\frac{\partial}{\partial t}F_n = F_{n+1} - F_{n-1} \qquad \textit{and} \qquad F_{n+1} = V_- F_n = F_n V_+,$$

where the very restricting conditions $V_-V_+ = 1$ and $\mathrm{rank}(1 - V_+V_-) = 1$ are assumed for the operators V_-, $V_+ \in \mathcal{L}(E)$. Although this ansatz has been found as a discretization of the corresponding continuous procedure, in [5] Bauhardt/Pöppe stress the fact that it is not possible to recover the latter simply as limiting case.

In contrast, our operator method is such a unified approach, which allows to treat as well the discrete as the continuous case with essentially the same formalism, see Subsection 2.8.

c) Pöppe's device immediately aims at scalar solutions. He does not investigate equations directly on the operator level.

2.8 Survey of results for other soliton equations

Next we shall give a survey of the results we obtained for other soliton equations so far. This includes as well continuous equations, which depend on two continuous space variables $x, y \in \mathbb{R}$, as lattice equations, where the space variable $n \in \mathbb{Z}$ is discrete.

Continuous equations: *Korteweg-de Vries equation (KdV), modified Korteweg-de Vries equation (mKdV), Kadomtsev-Petviashvili equation (KP), Boussinesq equation (B), sine-Gordon equation (sG), nonlinear Schrödinger equation (NLS).*

Lattice equations: *Wadati lattice (W), Langmuir lattice (L), Toda lattice (T).*

The results presented below are only partially published.

As already mentioned, our strategy was implemented for the first time in [3] for the Korteweg-de Vries equation. Subsequently, it was worked out in [52] for the sine-Gordon equation. The situation for Korteweg-de Vries/modified Korteweg-de Vries equation and Wadati/Langmuir lattice, which are linked by the (discrete) Miura transform and the continuum approximation, respectively, was clarified in [52] as well. For the Toda lattice, we refer to [53], and for the Kadomtsev-Petviashvili equation to [13].

Closely following the arguments outlined in Subsection 2.2, all statements can be transferred canonically. To avoid repetitions, here we restrict to a schematical representation of our results.

(1) Operator versions of equations

To begin with, in part **(1)** we state our operator-valued translations, namely a) the operator equation, b) its operator solution and the underlying base equations.

Continuous equations

a) Operator equations

KdV	$U_t = U_{xxx} + 3(UU_x + U_xU)$
mKdV	$U_t = U_{xxx} + 3(U^2U_x + U_xU^2)$
KP	$U_{xt} = \frac{1}{4}(U_{xxx} + 6U_x^2)_x + \frac{3}{4}U_{yy} + \frac{3}{2}[U_y, U_x]$
B	$U_{tt} = U_{xx} + U_{xxxx} + 3(U_xU_{xx} + U_{xx}U_x) - \mathrm{i}\sqrt{3}[U_t, U_x]$
sG	$\Big((1+U_-)^{-1}U_{-,x} + (1-U_+)^{-1}U_{+,x}\Big)_t = \frac{1}{2}\Big((1-U_+)^{-1}(1+U_-) - (1+U_-)^{-1}(1-U_+)\Big)$
NLS	$\mathrm{i}U_t + U_{xx} - 2U\overline{U}U = 0$ $\mathrm{i}\overline{U}_t - \overline{U}_{xx} + 2\overline{U}U\overline{U} = 0$

b) Operator solutions and base equations

KdV	$U=\Big((1+L)^{-1}(AL+LA)\Big)_x$	$L_x=AL, L_t=A^3L$
mKdV	$U=-\mathrm{i}(1-L^2)^{-1}(AL+LA)$	$L_x=AL, L_t=A^3L$
KP	$U=(1+L)^{-1}(BL+LA)$	$L_x=(A+B)L, L_y=(A^2-B^2)L,$ $L_t=(A^3+B^3)L$
B	$U=(1+L)^{-1}(A_+L+LA_-)$ with $A_\pm=A\pm\mathrm{i}A^{-1}\Omega/\sqrt{3}$	$L_x=AL, L_t=\Omega L$ for $[A,\Omega]=0, \Omega^2=A^2(1+A^2)$
sG	$U_\pm=(1\pm L)^{-1}(ALA^{-1}+L)$	$L_x=AL, L_t=A^{-1}L$
NLS	$U=(1-L\overline{L})^{-1}(AL+L\overline{A})$ $\overline{U}=(1-\overline{L}L)^{-1}(\overline{A}\,\overline{L}+\overline{L}A)$	$L_x=AL, L_t=\ \ \mathrm{i}A^2L$ $\overline{L}_x=\overline{A}\,\overline{L}, \overline{L}_t=-\mathrm{i}\overline{A}^2\overline{L}$

Lattice equations

a) Operator equations

T	$\Big((1+U_n)^{-1}U_{n,t}\Big)_t=$ $=(1+U_n)^{-1}(1+U_{n+1})-(1+U_{n-1})^{-1}(1+U_n)$
L	$U_{n,t}=(1+hU_n)U_{n+1}-U_{n-1}(1+hU_n)$
W	$U_{n,t}=(1+U_n^2)U_{n+1}-U_{n-1}(1+U_n^2)$

b) Operator solutions and base equations

T	$U_n=(1+L_n)^{-1}(VL_nV-L_n)$	$L_{n+1}=V^2L_n, L_{n,t}=(V-V^{-1})L_n$
L	$U_n=\frac{1}{h}V^{-1}(1+L_{n+1})^{-1}(VL_{n+1}V-L_{n+1})$ $-\frac{1}{h}V^{-1}(1+L_n)^{-1}(VL_nV-L_n)$	$L_{n+1}=VL_n, L_{n,t}=(V-V^{-1})L_n$
W	$U_n=-\mathrm{i}V^{-1}(1-L_n^2)^{-1}(VL_nV-L_n)$	$L_{n+1}=VL_n, L_{n,t}=(V-V^{-1})L_n$

In [6], Bauhardt and Pöppe succeeded in the treatment of the Zakharov-Shabat system (ZS system) on the operator level and deduced scalar solutions by trace methods. Nevertheless, for the ZS system a comfortable reformulation in terms of determinants is no longer possible. Their result is:

Proposition 2.28 *([6], Theorem 2.1) Let $F(x,t)$, $G(x,t)$ be families of operators defined on a suitable normed space, decaying sufficiently fast as $x\to-\infty$ so that the integrals mentioned below exist. Assume that F and G are trace class and $1-FG$ is nonsingular for all x and t.*

Moreover, let A be an invertible linear operator, F, G be holomorphic functions, and

$$F_x=AF,\ g(A)F_t\ =\ f(A)F,\quad G_x=AG,\ g(-A)G_t=-f(-A)G.$$

Then

$$\begin{aligned} R(x,t) &= (1-FG)^{-1}(AF+FA), \\ Q(x,t) &= (1-GF)^{-1}(AG+GA) \end{aligned}$$

are solutions of the following generalization of the ZS system to operators

$$g(\mathcal{L})\begin{pmatrix} R_t \\ Q_t \end{pmatrix} = f(\mathcal{L})\begin{pmatrix} R \\ -Q \end{pmatrix},$$

where $\mathcal{L}$ is given by the rule

$$\mathcal{L}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_x - R\int_{-\infty}^{x}(QU+VR)dx' - \int_{-\infty}^{x}(UQ+RV)dx'\,R \\ -V_x + Q\int_{-\infty}^{x}(UQ+RV)dx' - \int_{-\infty}^{x}(QU+VR)dx'\,Q \end{pmatrix}.$$

Remark 2.29 *Beside those direct interpretations, Blohm has recently given algorithms for the construction of operator solutions of certain hierarchies. For instance, in [8] a model is introduced, which allows to obtain the Toda lattice and its generalizations of higher order, and [9] is about a model for a hierarchy comprising the Korteweg-de Vries equation.*

Furthermore, a modification yields a model, which is closely related to the ZS system. In this model, example given, the nonlinear Schrödinger equation is translated into

$$U_t + U_{xx}E + 2U_x[U,E] = 0$$

with underlying base equations $L_x = -AELE$ and $L_t = -2A^2ELE$ (E an involution commuting with A).

An essential difficulty in the treatment of the ZS system is that one arrives at a system of two (coupled) scalar equations instead of a single equation. That means that one has to look for a substitute for the trace. This difficulty has been overcome by Blohm in [9] in the following manner:

Let F be a Banach module over $\mathbb{C}^{2\times 2}$, $c \in F$, and $a : F \to \mathbb{C}^{2\times 2}$ a module homomorphism, where we denote the evaluation of a at an element $x \in F$ with (x,a). As natural analogue for the concept of one-dimensional operators, Blohm uses module homomorphisms $a\triangle c : F \to F$, which are defined by

$$(a\triangle c)x = (x,a)c \quad \text{for } x \in F.$$

Since the rule $\mathrm{Tr} : a\triangle c \to (c,a)$ does not yield a well-defined mapping any more, one has to proceed more carefully. Provided that there is an $x_0 \in F$ with $(x_0,a) = 1 \in \mathbb{C}^{2\times 2}$, the rule

$$\mathrm{Tr}_{a,c}(Y(a\triangle c)) = (Yc,a)$$

yields a well-defined operator on $D(\mathrm{Tr}_{a,c}) = \{Y(a\triangle c) | Y \in L(F)\}$, which reproduces the property of a trace, i.e.

$$\mathrm{Tr}_{a,c}(X(a\triangle c)Y(a\triangle c)) = \mathrm{Tr}_{a,c}(Y(a\triangle c))\mathrm{Tr}_{a,c}(X(a\triangle c)).$$

Finally, it remains to mention that those equations, which can be solved by Blohm's ZS model, can be characterized by so-called zero-curvature conditions $Z_x - X_t = [X,Z]$. The latter is the compatibility condition of

$$\phi_x = X\phi \quad \text{and} \quad \phi_t = Z\phi$$

(ϕ is $\mathbb{C}^2$-valued). The first of these equations is just the scattering problem corresponding to the ZS system (with

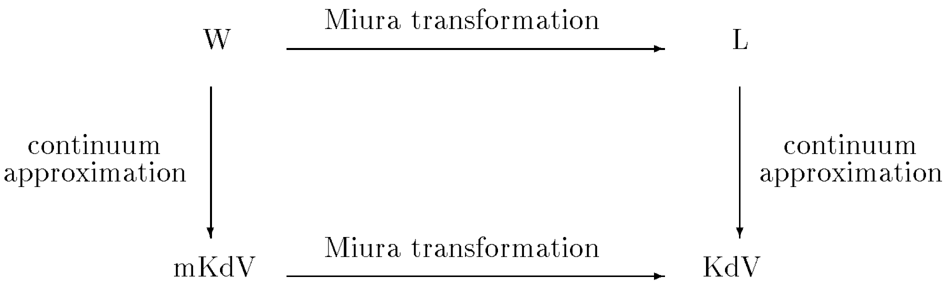
$$X = X(\lambda) = \lambda\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},$$

where λ is the eigenvalue and r, q the two functions appearing in the evolution equation later on), whereas the second assigns the time behaviour.

Starting from the ansatz that Z is a polynomial in λ, the coefficient-matrices can be explicitly expressed in terms of $\mathrm{Tr}_{a,c}(U_j)$, U_j the operator solutions of the operator hierarchy in Blohm's model, and then the evolution equation corresponding to the zero-curvature condition can be determined.

(For details we refer to [9]).

It is worth mentioning that also other important relations between soliton equations remain valid on the operator level. The following convincing situation has been discussed in [52], Subsection 1.4.



Proposition 2.30 *The Miura transformation, which is defined by $M(U) := U^2 + \mathrm{i}U_x$, translates solutions U of the mKdV into solutions $M(U)$ of the KdV. Analogously, the discrete Miura transformation $M(U_n) := \frac{1}{h}\big((1-\mathrm{i}U_n)(1+\mathrm{i}U_{n+1})-1\big)$ translates solutions U_n of the Wadati lattice into solutions $M(U_n)$ of the Langmuir lattice.*

The special solutions stated in the tables, are translated into each other as well.

Proposition 2.31 *According to the rule $U(x,t) := \frac{1}{h}U_n(\frac{3}{h^3}t)$, $nh = x - \frac{6}{h^2}t$, in the limit $h \to 0$ the Langmuir lattice is translated into the KdV and the Wadati lattice into the mKdV (continuum approximation) .*

On condition that $V = \exp(hA)$, the special solutions stated in the tables are also translated into each other.

(2) Explicit solution formulas

Next we give a list of the solution formulas due to our operator method.

Continuous Equations

KdV	$u = 2\big(\log\delta(1+L)\big)_{xx}$
mKdV	$u = \mathrm{i}\big((\log\dfrac{\delta(1-L)}{\delta(1+L)}\big)_x$
KP	$u = \big((\log\delta(1+L)\big)_x$
B	$u = 2\big((\log\delta(1+L)\big)_x$
sG	$u_\pm = \mp\dfrac{\delta(1\mp L)}{\delta(1\pm L)} \pm 1$
NLS	$u\overline{u} = -\big((\log\delta(1-L\overline{L})\big)_{xx}$

Lattice equations

T	$u_n = \dfrac{\delta(1+L_{n+1})}{\delta(1+L_n)} - 1$
L	$u_n = \dfrac{1}{h}\big(\log\dfrac{\delta(1+L_{n+1})}{\delta(1+L_n)}\big)_t$
W	$u_n = \dfrac{\mathrm{i}}{2}\big(\log\dfrac{\delta(1-L_n)}{\delta(1+L_n)}\big)_t$

Remark 2.32 *a) Solutions of the sine-Gordon equation* $u_{xt} = \sin(u)$ *arise from the version treated above using the transformation* $u = \mathrm{i}\log(1+u_-)/(1-u_+)$ *(for a motivation confer [52]). Thus, as solution formula, we get* $u = 2\mathrm{i}\log\left(\delta(1+L)/\delta(1-L)\right)$.

b) Solutions of the Toda lattice $\widehat{u}_{n,tt} = \exp(-(\widehat{u}_n - \widehat{u}_{n-1})) - \exp(-(\widehat{u}_{n+1} - \widehat{u}_n))$ *arise from the above form using the transformation* $\widehat{u}_n = -\log(1+u_n)$*, and, as a result, we get* $\widehat{u}_n = -\log\left(\delta(1+L_{n+1})/\delta(1+L_n)\right)$.

Remark 2.33 *In the working group 'Gravitationstheorie' of Neugebauer and Meinel, Institut of Theoretical Physics in Jena, considerable results for the Ernst equation*

$$\Re(f)\Delta f = (\nabla f)^2,$$

$f = f(\rho,\zeta)$ *the (complex) Ernst potential, which is the reduction of the vacuum Einstein's field equations to the stationary axisymmetric situation, have been achieved. Although this soliton equation is physically of great importance in connection with astro physics, mathematically it has aroused astonishingly little attention.*

Thus, it is remarkable that in [38], [41], and [42], they succeeded in finding analogies to the usual theory of soliton equations (N-solitons, elliptic solutions, inverse scattering transform). In the articles, it turned out to be a particular subtle point to sort out the physically relevant among the found solutions. Since our method enables us especially to investigate the qualitative behaviour of solutions (see Subsection 3.1), it seems to us to be an attractive project to consider the Ernst equation from our point of view.

3 Construction and investigation of solution classes

In the following section we study applications of our solution formula (3). We describe solution classes, which are obtained for special choices of the 'generating operator' A and examine how the geometry of the solutions can be understood in terms of the functional analytic (or algebraic) properties of A.

As motivation we remark that inserting diagonal matrices (of finite dimension) yields the well-known soliton solutions of Hirota:

Lemma 3.1 *Let* $a, c \in \mathbb{C}^N$ *be arbitrary vectors. Set*

$$A = \begin{pmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & k_N \end{pmatrix},$$

where $k_j \in \mathbb{C}$ *are pairwise different and* $k_i + k_j \neq 0\ \forall i,j = 1,\dots,N$.

Then the solution of the KdV computed according to Theorem 2.6 and Corollary 2.12 equals $u(x,t) = 2\partial_x^2 \log p(x,t)$,

$$\begin{aligned} p(x,t) &= \det\left(\left(\delta_{ij} + \exp\left(k_i x + k_i^3 t\right)\frac{a_j c_i}{k_i + k_j}\right)_{i,j=1}^N\right) \\ &= 1 + \sum_{n=1}^N \sum_{i_1<\dots<i_n} \prod_{j=1}^n \exp\left(k_{i_j}x + k_{i_j}^3 t + \delta_{i_j}\right) \prod_{j'=j+1}^n \left(\frac{k_{i_j} - k_{i_{j'}}}{k_{i_j} + k_{i_{j'}}}\right)^2 \text{ with } \exp(\delta_j) = \frac{a_j c_j}{2k_j}. \end{aligned}$$

For $0 < k_1 < \dots < k_N$ *and* $\delta_j \in \mathbb{R}$ *this leads precisely to the N-soliton solutions of Hirota (confer Proposition 2.23).*

As a generalization, we shall derive two solution classes:

- For arbitrary (finite-dimensional) matrices we obtain the so-called negatons. Roughly speaking, negatons can be visualized as superpositions of several solitary waves which come in groups. Their asymptotic behaviour is encoded in the Jordan canonical form of A.
- Diagonal operators A lead to solutions which are superpositions of a countable number of independently travelling solitary waves. Boundedness properties of A nicely reflect the convergence of the above superposition.

3.1 Asymptotic behaviour of negatons

In the beginning of the 1990s, Matveev and Salle [33] systematically developed the formalism of Darboux transformations and derived solution formulas for soliton equations containing Wronskians.

For the KdV (see [34]) they were led to

$$u(x,t) = u_0 + 2\frac{\partial^2}{\partial x^2}\log W \qquad \text{with } W = W(\varphi_1,\ldots,\varphi_n) = \det\Big(\Big(\frac{\partial^{i-1}}{\partial x^{i-1}}\varphi_j\Big)_{i,j=1}^{n}\Big), \tag{10}$$

where $\varphi_i = \varphi_i(x,t)$ are fixed, linear independent solutions of the system

$$L\varphi = \lambda\varphi \text{ and } \varphi_t = A\varphi, \qquad \begin{aligned} L &= \partial_x^2 + u_0(x,t), \\ A &= -\Big(4\partial_x^3 + 6u_0(x,t)\partial_x + 3u_{0,x}(x,t)\Big) \end{aligned}$$

(with eventually different eigenvalues λ).

By analogous choices these formulas yield two important solution classes, which are called positons and negatons in the literature:

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log W \qquad \text{with } W = W(\varphi,\ldots,\frac{\partial^n}{\partial k^n}\varphi), \tag{11}$$

where $\varphi = \varphi(k;x,t)$ is defined as

$$\varphi_{\text{pos}} = \sin\Big(k(x - k^2 t + \delta(k))/2\Big) \qquad \text{(positon of order } n\text{) and}$$

$$\varphi_{\text{neg}} = \begin{cases} \cosh\Big(k(x + k^2 t + \delta(k))/2\Big) \\ \sinh\Big(k(x + k^2 t + \delta(k))/2\Big) \end{cases} \qquad \text{(negaton of order } n\text{)},$$

respectively, (observe $\lambda = k^2/4$). In order to construct superpositions of a finite number N of positons or negatons of arbitrary orders $n_1,\ldots,n_N$, the Wronsky formula (11) with

$$W = W(\varphi_1(k_1),..,\frac{\partial^{n_1}}{\partial {k_1}^{n_1}}\varphi_1(k_1),......,\varphi_N(k_N),..,\frac{\partial^{n_N}}{\partial {k_N}^{n_N}}\varphi_N(k_N))$$

where $\varphi_j \in \{\varphi_{\text{pos}},\varphi_{\text{neg}}\}$, is employed. Note that, in order to generate a negaton, it is necessary to make an appropriate choice between the two possibilities for φ_{neg} (confer [35] and addititonaly Remark 3.2).

Remark 3.2 *The N-solitons are obtained from $W(\varphi_1,\dots,\varphi_N)$ by setting*

$$\varphi_j(x,t)=\begin{cases}\cosh\Big(k_j(x+k_j^2t+\delta(k_j))/2\Big), & j \text{ odd},\\ \sinh\Big(k_j(x+k_j^2t+\delta(k_j))/2\Big), & j \text{ even}.\end{cases}$$

Remark 3.3 *a) Only positons of odd order provide interesting features: they turn out to be stable objects. In case of even order, periodic solutions with singularities are obtained. For example for order zero, $u(x,t)=-(k^2/2)\sin^{-2}\left(k(x-k^2t+\delta)/2\right)$ (compare with the discussion in [35]).*

b) The notion of negatons extends the well-known notion of solitons: solitons are negatons of order zero.

Matveev and his colaborators have concentrated on the study of positons. In [36], [37] they described the asymptotic behaviour of solutions which are superpositions of an arbitrary number of solitons and positons of order 1. For negatons, similar results could be proved by Rasinariu et al. [51], but only in very particular cases (for example negatons of order $n\le 4$ or the collision of two negatons of order 1). Both for positons and negatons the above mentioned authors formulated expectations for the remaining open general case.

The results of this section contain a complete treatment of the general case for negatons. They affirm the expectations of Rasinariu et al., which correspond precisely to Matveev's predictions for positons.

Within our operator method, negatons arise by inserting (finite) matrices A into the solution formula (3). The following lemma shows that we can always suppose A to be in Jordan canonical form.

Lemma 3.4 *([3], [52] Lemma 4.0.1) Let A be an arbitrary matrix with $0\notin\operatorname{spec}(A)+\operatorname{spec}(A)$ and $u(x,t)=2\partial_x^2\log\det\left(1+\exp(Ax+A^3t)\Phi_{A,A}^{-1}(a\otimes c)\right)$ the corresponding solution of the KdV according to Theorem 2.6, Corollary 2.12.*

Then there are vectors $\hat a$, $\hat c$ such that

$$u(x,t)=2\partial_x^2\log\det\Big(1+\exp(J_Ax+J_A^3t)\Phi_{J_A,J_A}^{-1}(\hat a\otimes\hat c)\Big),$$

where J_A is the Jordan canonical form of A.

Let us place a non-degeneracy assumption.

Assumption 3.5 *Suppose that A is a Jordan matrix which contains N Jordan blocks A_j of dimension n_j corresponding to the eigenvalues k_j, i.e.*

$$A=\begin{pmatrix}A_1 & 0 & \cdots & 0\\ 0 & A_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & A_N\end{pmatrix}\quad \textit{where } A_j=\begin{pmatrix}k_j & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & k_j\end{pmatrix},$$

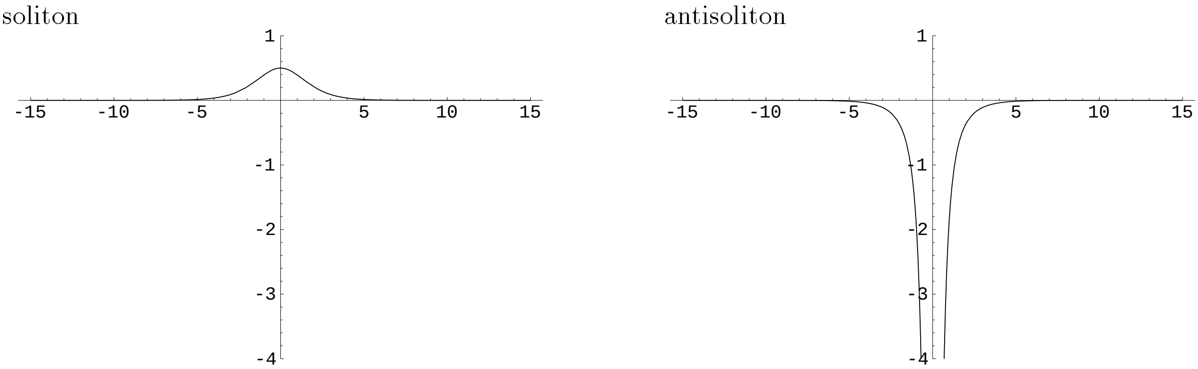
with pairwise different eigenvalues $k_j\in\mathbb{C}$ satisfying $k_i+k_j\neq 0$ $(\forall i,j=1,\dots,N)$.

As we shall focus on real solutions, we suppose in addition that the vectors a, c, and the eigenvalues are real. A closer examination of the formulas shows that we may even assume

$$k_N>\ldots>k_1>0.$$

In what follows, we try to explain how the asymptotic behaviour can be read from the algebraic properties of the matrix A.

To this end we first take a look at the elementary building blocks of negatons, namely negatons of order zero. Their shape depends on the sign of the parameters a_1, c_1. Setting $(a_1c_1)/(2k_1) = \epsilon_1 \exp(\delta_1)$ with a real parameter δ_1 and a sign ϵ_1, we distinguish between
a) 'regular' solitons $u(x,t) = (k_1^2/2)\cosh^{-2}(k_1(x + k_1^2 t + \delta_1)/2)$ for $\epsilon_1 = 1$ and
b) 'singular' solitons $u(x,t) = -(k_1^2/2)\sinh^{-2}(k_1(x + k_1^2 t + \delta_1)/2)$ for $\epsilon_1 = -1$
(In the literature they are also called solitons and antisolitons).



With regard to the fact that negatons may have poles, it is useful to provide an appropriate notion of convergence.

For any fixed t, $u_t(\cdot) := u(\cdot, t)$ is the restriction of a meromorphic function defined on an appropriate neighbourhood U (eventually $U = U(t)$) of the real axis. This means that $u_t(\cdot)$ can also be viewed as a mapping to the Riemann number sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We equip $\hat{\mathbb{C}}$ with some metric, say the cordal metric $\mathrm{d}^{\mathrm{cord}}$, which is given by

$$\mathrm{d}^{\mathrm{cord}}(w,z) = |\pi^{-1}(w) - \pi^{-1}(z)|, \qquad \pi^{-1} : \hat{\mathbb{C}} \to S^2 \subseteq \mathbb{R}^3,$$

where π denotes the stereographic projection $\pi : S^2 \to \hat{\mathbb{C}}$. With these conventions, we can ask for the uniform convergence of a family $u_t : \mathbb{R} \to \hat{\mathbb{C}}$ as $t \to \pm\infty$.

Definition 3.6 *We say that two functions $u(x,t)$ and $v(x,t)$ have the same asymptotic behaviour as $t \to \infty$ ($t \to -\infty$), briefly*

$$u(x,t) \approx v(x,t) \qquad \text{as } t \approx \infty \ (t \approx -\infty), \tag{12}$$

if, for every $\epsilon > 0$, there is t_ϵ such that for $t > t_\epsilon$ ($t < t_\epsilon$) the estimate $\mathrm{d}^{\mathrm{cord}}(u(x,t), v(x,t)) < \epsilon$ is satisfied for all $x \in \mathbb{R}$.

Now we are in position to formulate our main result. The proofs are based on methods developed by the second author in her thesis [52]. The formulation chosen below takes into account certain improvements which she obtained in [54] in the context of the sine-Gordon equation.

Theorem 3.7 *([52] Theorem 4.3.1, [54] Theorem B) To data as in Assumption 3.5 we associate the curves*

$$\Gamma_{j,m_j}(x,t) = k_j x + k_j^3 t + \log \mid \tau \mid^{\mp m_j} + (\delta_j + \delta_j^{\pm} + \delta_{j,m_j}^{\pm}) \tag{13}$$

$$\text{for } m_j = -(n_j-1), -(n_j-1)+2, \ldots, (n_j-1)-2, (n_j-1) \tag{14}$$

and, along these curves, the solitons

$$u_{j,m_j}^{\pm}(x,t) \;=\; 2k_j^2 \ell_j^{\pm}(1+\ell_j^{\pm})^{-2} \quad \text{with } \ell_j^{\pm}(x,t) = (-1)^{\frac{(n_j-1)+m_j}{2}} \epsilon_j \exp\Big(\Gamma_{j,m_j}(x,t)\Big). \tag{15}$$

Here we suppose that the vector a (and c accordingly) is decomposed as $a = (a_1, \ldots, a_N)^t$ with $a_j = (a_j^{(1)}, \ldots, a_j^{(n_j)})^t$, corresponding to the Jordan canonical form of A, and that $\delta_j \in \mathbb{R}$ and $\epsilon_j = \pm 1$ are defined by $a_j^{(1)} c_j^{(n_j)}/(2k_j)^{n_j} = \epsilon_j \exp(\delta_j)$.

The quantities $\delta_j^{\pm}$, $\delta_{j,m_j}^{\pm}$ represent the phase shifts, which arise in the asymptotic expression, and we have

$$\exp(\delta_j^{-}) = \prod_{j'=1}^{j-1} \left[\frac{k_{j'}-k_j}{k_{j'}+k_j}\right]^{2n_{j'}} \text{ and } \exp(\delta_j^{+}) = \prod_{j'=j+1}^{N} \left[\frac{k_{j'}-k_j}{k_{j'}+k_j}\right]^{2n_{j'}}, \text{ resp.,} \tag{16}$$

$$\text{and } \exp(\delta_{j,m_j}^{\pm}) = (4k_j^3)^{\mp m_j} \frac{\left(\frac{(n_j-1)\pm m_j}{2}\right)!}{\left(\frac{(n_j-1)\mp m_j}{2}\right)!}. \tag{17}$$

Then the asymptotic behaviour of the solution $u(x,t)$ of the KdV obtained from Theorem 2.6 and Corollary 2.12 is given by

$$u(x,t) \approx \sum_{j=1}^{N} \sum_{m_j} u_{j,m_j}^{\pm}(x,t) \qquad \text{for } t \approx \pm\infty \tag{18}$$

(the summation indices m_j being determined by (14)).

Qualitatively the content of Theorem 3.7 can be summarized as follows (see [52], immediately after Theorem 4.3.1):

Interpretation

Negatons are characterized by the finite set of eigenvalues $k_1, \ldots, k_N$. Here we have supposed them to be of geometric multiplicity one to avoid the discussion about cancellation phenomena.

a) A single negaton of order n belongs to a real eigenvalue k of algebraic multiplicity n. Without loss of generality we assume $k > 0$. Such a negaton is a wave packet consisting of n regular and singular solitons as members. Their shapes are identic and depend only on k.

The main observation is the following: the geometric center of the wave packet moves with constant velocity $-k^2$, but its members drift away from each other in such a way that the distance between one of them and the geometric center itself increases at most logarithmically.

Roughly speaking, each soliton starts for $t << 0$ on one side of the center and approaches the center with logarithmic speed. Sometime it changes to the other side of the center and then, for $t >> 0$, it moves away from the center again with logarithmic speed. According to *this interpretation, which is also confirmed by computer experiments, the solitons appear in the asymptotic forms for $-\infty$ and $+\infty$ precisely in reversed order.*

Moreover, we can read from the formulas that regular and singular solitons come alternately. In particular there are only two types of asymptotic forms, and it depends only on the sign ϵ which type occurs.

The wave packet as a whole, more precisely the trajectory of its center, does not suffer a phase shift by the internous collisions of its members.

b) In general, given N eigenvalues $k_1, \ldots, k_n$ of algebraic multiplicity $n_1, \ldots, n_N$, the negaton consists of N wave packets as in a) and represents the superposition of N negatons of orders $n_1, \ldots, n_N$.

These meet – similarly as N-solitons – in elastic collisions without changing their shape but suffering a phase shift. The latter is explicitly given by (16).

It is worth mentioning that (16) is a natural generalization of the corresponding formula for N-solitons, the only difference is the appearance of the algebraic multiplicities in the exponents.

c) N-solitons are recovered in the class of negatons by taking all algebraic multiplicities $n_j = 1$ and all parameters $\epsilon_j = 1$.

Corollary 3.8 *([52], Korollar 4.3.2)*

The sum of the phase shifts vanishes: $\displaystyle\sum_{j=1}^{N} n_j(\delta_j^+ - \delta_j^-) = 0.$

Remark 3.9 *a) A similar statement was proved in the recent work [54] for the sine-Gordon equation, where even singularity-free, hence physically relevant solutions are obtained.*

Furthermore, a new phenomenon comes into the game, namely breathers, which correspond to pairs of complex conjugated eigenvalues. Therefore, the structure of negatons is much richer and, of course, the asymptotic analysis gets more complicated.

b) It is worth noting that the properties of negatons, as discussed above, are quite different from those of positons: positons are weakly localized, decay like $1/x$, and oscillate for x large. In addition they collide without phase shift (see [37]).

For sake of illustration, we have gathered some computer graphics in the appendix.

3.2 Countable superpositions of solitons

It is a fundamental fact that single solitons can be combined to N-solitons by 'nonlinear superposition'. Hence it is suggestive to look for solutions which are obtained from N-solitons as an appropriate limit $N \to \infty$, in other words by a superposition of countably many solitons.

In the context of our method the access to these solutions relies on the freedom of choosing the Banach space E in Theorem 2.6. Naturally generalizing the hypothesis of Lemma 3.1, we now suppose:

Assumption 3.10 *Let E be one of the classical sequence spaces c_0 or (a weighted) l_p, $1 \leq p < \infty$, and A the diagonal operator on E which is defined by a bounded sequence $k = (k_i)_i \in \ell_\infty$, i.e.*

$$A : E \longrightarrow E, \quad A(\xi_i)_i = (k_i \xi_i)_i.$$

First we concentrate on the case $0 \notin \mathrm{spec}(A) + \mathrm{spec}(A)$. Then the operator X which has to meet the condition $\mathrm{rank}(AX + XA) = 1$ is given by Corollary 2.12. As a result we obtain a solution class, which can be expressed by any determinant.

Proposition 3.11 *([3] Proposition IV C 3 (i)) In addition to Assumption 3.10 suppose* $\inf_{i,j} |k_i + k_j| > 0$. *Then, for every* $a = (a_i)_i \in E'$, $c = (c_i)_i \in E$, *the operator*

$$L(x,t) = \left(\frac{a_j c_i}{k_i + k_j} \exp\left(k_i x + k_i^3 t\right) \right)_{i,j=1}^{\infty}$$

belongs to the component $\mathcal{A}(E)$ *of every given quasi-Banachideal* $\mathcal{A}$ *with continuous determinant* δ, *and*

$$u = 2\frac{\partial^2}{\partial x^2} \log \delta(1 + L)$$

is a solution of the KdV, whenever the argument of the logarithm is non-vanishing.

Next we ask the question whether we can drop this condition and admit diagonal operators A with $0 \in \mathrm{spec}(A) + \mathrm{spec}(A)$.

Of course, now Corollary 2.12 is no longer applicable. Thus we have to face the problem that for the formal solution $X = \left(\frac{a_j c_i}{k_i + k_j}\right)_{i,j}$ of the operator equation $AX + XA = a \otimes c$ (for $a \in E'$, $c \in E$) even boundedness is not clear a priori. A reasonable ansatz, in order to guarantee that X belong to a quasi-Banach ideal with a well-behaved determinant, is to prescribe additional assumptions for the choice of $a \in E$, $c \in E'$ (in other words for the one-dimensional operator $a \otimes c$). The following proposition tells that this is feasible without damage to the quality of the solution formulas.

For sake of simplicity we only consider sequences $k = (k_i)_i$ with positive entries $k_i > 0\ \forall i$.

Proposition 3.12 *([3] Proposition IV C 3 (ii)) Suppose Assumption 3.10. If* $k_i > 0$ *for all* i, *then the operator*

$$L(x,t) = \left(\frac{a_j c_i}{k_i + k_j} \exp\left(k_i x + k_i^3 t\right) \right)_{i,j=1}^{\infty}$$

for $(a_i/\sqrt{k_i})_i \in E'$ *and* $(c_i/\sqrt{k_i})_i \in E$ *belongs to the component* $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1(E)$ *of the quasi-Banach ideal* $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1$, *and*

$$u = 2\frac{\partial^2}{\partial x^2} \log \det{}_\lambda(I + L)$$

($\det_\lambda$ *denotes the spectral determinant on* $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1$) *is a solution of the KdV provided that the argument of the logarithm is non-vanishing.*

Remark 3.13 *The quasi-Banach ideal* $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1 = \bigcup_{E,F} \mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1(E,F)$ *consists of operators* $T \in \mathcal{L}(E,F)$ *(E, F Banach spaces) which factorize through an* L_1*-space, a Hilbert space, and finally an* L_∞*-space (with the usual product norm). We have* $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1 \subseteq \mathcal{S}_1^{eig}$, *a fact which relies essentially on Grothendieck's theorem (see Pisier [48]). The existence of the spectral determinant* $\det_\lambda$ *then follows from Proposition 2.7.*

In papers of Gesztesy et al. (see [21] for the KdV), a completely different approach to the construction of 'limit-solitons' (in the notation of the authors) was developed.

Point of departure is the N-soliton in the form

$$u_N(x,t) = 2\frac{\partial^2}{\partial x^2}\log\det\Big(1_N + C_N(x,t)\Big)$$
$$\text{with } C_N(x,t) = \left(\frac{b_i b_j}{k_i+k_j}\exp\frac{1}{2}\Big((k_i+k_j)x + (k_i^3+k_j^3)t\Big)\right)_{i,j}^N$$

where $k_i > 0$, $b_i > 0$. This agrees with the notations in [21] except of transforming to the KdV as in (1) and rescaling of the k_i, b_i (namely $\kappa_i = k_i/2$ for the parameter κ_i). The coincidence with the representation in Lemma 3.1 easily follows from the general properties of a determinant. Here $C_N(x,t)$ is regarded as an operator on ℓ_2 in the usual way.

Main assumption

a) $C_N(x,t)$ converges for $N \to \infty$ with respect to the trace norm $\|\cdot|\mathcal{N}\|$ to an operator $C_\infty(x,t)$, which therefore is a trace class operator, $C_\infty(x,t) \in \mathcal{N}(\ell_2)$.

b) The Schrödinger operators $H_N(t) = d^2/dx^2 + u_N(x,t)$, which appear in the scattering problem, are uniformly bounded on $L_2(\mathbb{R})$ with respect to N (and t).

This implies the existence of the limit-soliton

$$u_\infty(x,t) = 2\frac{\partial^2}{\partial x^2}\log\det\Big(1 + C_\infty(x,t)\Big) \tag{19}$$
$$\text{with } C_\infty(x,t) = \left(\frac{b_i b_j}{k_i+k_j}\exp\frac{1}{2}\Big((k_i+k_j)x + (k_i^3+k_j^3)t\Big)\right)_{i,j}^\infty,$$

where det denotes the Fredholm determinant.

The main result of Gesztesy et al., which is based on a careful examination of the spectral properties of the Schrödinger operators H_N (corresponding to N-solitons) and of its scattering data for $N \to \infty$, is:

Proposition 3.14 *([21] Theorem 6.1) Let $(k_i)_i \in \ell_\infty$ be a bounded sequence with positive, pairwise different entries k_i and $(b_i)_i$ a positive sequence such that $(b_i^2/k_i)_i \in \ell_1$.*

Then (19) is a solution of the KdV.

Propositions 3.11 and 3.12 allow to realize solutions of this type in an essentially broader context. In particular, (19) can be recovered in the following way:

Corollary 3.15 *(of Proposition 3.12) Let $(k_i)_i$, $(b_i)_i$ be sequences with positive entries such that $(k_i)_i \in \ell_\infty$, $(b_i^2/k_i)_i \in \ell_1$. Then*

$$L(x,t) = \left(\frac{b_i b_j}{k_i+k_j}\exp\frac{1}{2}\Big((k_i+k_j)x + (k_i^3+k_j^3)t\Big)\right)_{i,j}^\infty \in \mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1(\ell_2)$$

is a trace class operator on ℓ_2, i.e. $L \in \mathcal{N}(\ell_2)$ (note that we here use $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1(\ell_2) \subseteq \mathcal{N}(\ell_2)$, see [29]), and

$$u = 2\frac{\partial^2}{\partial x^2}\log\det(1+L)$$

is a (well-defined since $\det(1+L) > 0$*) solution of the KdV (*det *being the Fredholm determinant on $\mathcal{N}(\ell_2)$).*

This is exactly the solution class described by Gesztesy et al. (confer Proposition 3.14).

Remark 3.16 *Requiring in addition $(k_i)_i \in \ell_1$, Gesztesy et al. could even show that the limit-soliton (19) is a reflection-free potential (see [21] for the details).*

Remark 3.17 *a) The article [21] of Gesztesy et al. contains further important contributions, which we did not discuss and which are not accessible by our methods. For example the detailed description of the spectral properties of the Schrödinger operators*

$$H_\infty(t) = \frac{d^2}{dx^2} + u_\infty(x,t) \quad \textit{corresponding to a limit soliton } u_\infty(x,t),$$

leads to the solution of the following problem ([21] Theorem 5.9):

> *Given a countable, bounded set $\{k_j^2/4\}_j \subseteq (0,\infty)$, construct explicitly a (smooth, real-valued) potential u such that the point spectrum of $H = d^2/dx^2 + u$ contains the set $\{k_j^2/4\}_j$ and the continuous spectrum equals $(-\infty, 0]$.*

On the other hand, the involved analysis concerning the scattering data is not indispensable, if one is only interested in verification of the solution formulas. Here our approach turns out to be relatively simple.

b) In [22], Gesztesy and Renger were able to extend their methods to the Toda lattice. Here the examination of the Jacobi operator on $\ell_2(\mathbb{Z})$, which corresponds to the discrete scattering problem, becomes essentially. We explained in [53] how to obtain the limit-solitons of the Toda lattice by our methods.

To conclude the section, we show how to reduce the solution classes of Propositions 3.11 and 3.12 to a very special situation. This means:

- the reduction to the ideal component $\mathcal{N}(\ell_1)$ and
- the restriction to one-dimensional operators of the form $e_0 \otimes d$ with $e_0 = (1,1,\dots) \in \ell_\infty$ and d a sequence satisfying special summation properties. This way, the number of involved parameters can be reduced to the half.

Hence the determinant on $\mathcal{N}(\ell_1)$ is in some sense universal for the construction of solutions by means of diagonal operators.

Theorem 3.18 *([3] Theorem IV C 5)*
Suppose one of the following:
a) $\inf_{i,j} |k_i + k_j| > 0$ *and* $d = (d_i)_i \in l_1$,
b) $k_i > 0 \ \forall i$ *and* $d = (d_i)_i$ *with* $(d_i/k_i)_i \in l_1$.
Then the operator

$$L(x,t) = \left(\frac{d_i}{k_i + k_j} \exp\left(k_i x + k_i^3 t\right)\right)_{i,j=1}^{\infty} \tag{20}$$

belongs to the nuclear component $\mathcal{N}(\ell_1)$, and $u = 2\partial_x^2 \log \det_{\mathcal{N}}(I+L)$ solves the KdV whenever the argument of the logarithm is non-vanishing.

Moreover, each solution of Propositions 3.11 or 3.12 (corresponding to the cases a) and b), respectively) can be written in the form (20).

Remark 3.19 *The reduction of the two solution classes of Propositions 3.11 and 3.12 to the ideal component $\mathcal{N}(\ell_1)$, which we here obtained for the KdV, can be carried over in similar fashion to other soliton equations.*

The fact that, for the KdV, this reduction can be achieved for both solution classes by the same operator (20), relies on the special rank condition, which is given by the elementary expression $\Phi_{A,A}(X) = AX + XA$.

For instance for the Kadomtsev-Petviashvili equation, where the involved elementary expression is $\Phi_{A_1,A_2}(X) = A_2X + XA_1$, such a uniform reduction is only possible under the additional assumption

$$\left(\sqrt{k_i^{(2)}/k_i^{(1)}}\right)_i \in \ell_\infty.$$

Here A_1, A_2 are diagonal operators associated to the sequences $k^{(1)} = (k_i^{(1)})_i$, $k^{(2)} = (k_i^{(2)})_i$ (bounded, with positive entries).

The reader may find a detailed discussion in [13].

4 Some further aspects

Finally we shall examine some further questions related to our operator method.

The first, quite natural question asks for generalizations of the operator method in view of additional applications.

Because of the key role of the generating operator A (as underlined by the applications in Section 3), it is suggestive to enlarge the class of admissible operators A.

In the first subsection we shall sketch how the theory of C_0-semigroups allows to integrate unbounded operators A into the strategy. Some examples shall illustrate the difference to what was obtained before.

Another important question concerns the scope of the operator method. More precisely, we are interested to which extent classical solution techniques in soliton theory are compatible with our approach.

As it is well-known, the inverse scattering transform (IST) is a powerful tool for the general solution of the initial value problem

$$u_t = u_{xxx} + 6uu_x, \qquad u(x,0) = u_0(x),$$

for rapidly decreasing potentials $u_0(x)$. In the second subsection we shall explain how solutions obtained by IST can be recovered by our strategy.

4.1 Extension to unbounded operators

In Proposition 2.2 we have explained how one can get solutions of the operator KdV (2) from bounded operators $A \in \mathcal{L}(F)$ on a Banach space F. Let us reformulate the result in more abstract terms:

Proposition 4.1 *Let $A \in \mathcal{L}(F)$. Furthermore, let $L(x,t)$, $M(x,t)$ be two families of bounded operators, $\mathcal{C}^1$ in the time variable t and $\mathcal{C}^4$ in the space variable x, fulfilling the following properties:*

a) $L_x = AL$, $L_t = A^3L$ and $M_x = AM$, $M_t = A^3M$ (base equations),

b) $AL + LA = M$ (compatibility condition).

Then $U := \left((1+L)^{-1}M\right)_x$ is a solution of the operator KdV (2) in $\mathcal{L}$ whenever $(1+L)^{-1}$ exists.

As proved in [12], solutions of the operator KdV (2) for unbounded operators $A \in L(F)$ can be obtained in completely analogous fashion.

Proposition 4.2 *([12] Proposition 2.1) Let $A \in L(F)$ be densely defined and closed. Furthermore, let $L(x,t)$, $M(x,t)$ be families of bounded operators, $\mathcal{C}^1$ in the time variable t and $\mathcal{C}^4$ in the space variable x, as well as $L(x,t)f$, $M(x,t)f \in D(A^n)$ $(1 \le n \le 4)$ for all $f \in F$, fulfilling the following properties:*

a) $L_x = AL$, $L_t = A^3 L$ and $M_x = AM$, $M_t = A^3 M$ (base equations),

b) $ALf + LAf = Mf$ $\forall f \in D(A)$ (compatibility condition).

Then again $U := (1+L)^{-1}M)_x$ is a solution of the operator KdV (2) in $\mathcal{L}$ whenever $(1+L)$ is invertible.

Now the techniques of Section 2.2 are applicable without restriction, that is on the assumption that $M(x,t) = c \otimes m(x,t)$ one can extract scalar solutions of the KdV (1) by using the trace.

In the concrete realization of our strategy, we encounter two major difficulties.

(1) In order to solve the base equation $T_x = AT$, we could employ, for bounded operators $A \in \mathcal{L}(F)$, the exponential

$$T(x) = \exp(xA) = \sum_{n=0}^{\infty} \frac{x^n}{n!} A^n, \qquad A \in \mathcal{L}(F).$$

For unbounded operators $A \in L(F)$, the theory of semigroups (for a detailed introduction in this topic see [40], [43]) provides the substitute for this. Recall that a semigroup $(T(x))_{x \ge 0}$ of bounded operators on F is called a C_0-semigroup if it is strongly continuous, that is if $\lim_{x \to 0} T(x)f = f$ $\forall f \in F$.

By the infinitesimal generator A of a semigroup $(T(x))_{x \ge 0}$ one denotes the operator A defined on $D(A) = \{f \in F : \exists \lim_{x \to 0} (T(x)f - f)/x\}$ by $Af = \lim_{x \to 0} (T(x)f - f)/x$. In general, $A \in L(F)$. One easily observes that A is densely defined and closed.

Semigroups model perfectly the properties of the exponential, since $T(x)f \in D(A)$ for all $f \in D(A)$, and

$$\frac{d}{dx} T(x)f = AT(x)f = T(x)Af$$

(therefore, one symbolically writes $(e^{xA})_{x \ge 0}$).

Lemma 4.3 *([12] Theorem 2.3) Let $L(x,t) \in \mathcal{A}$, $\mathcal{A}$ a quasi-Banach ideal with a continuous determinant δ. Under the additional assumption that there is a constant $\lambda \in \mathbb{C}^\times$ such that λA generates a C_0-semigroup, the solution formula of Theorem 4.2 can be written as $u = 2\partial_x^2 \log \delta(1+L)$.*

(2) To attribute a meaning to the operator equation $AX + XA = C$ for a given operator $C \in \mathcal{K}(F)$, we proceed as follows: If $A \in L(F)$ generates a C_0-semigroup $(T(x))_{x \ge 0}$, then $(\mathcal{T}(x))_{x \ge 0}$, where $\mathcal{T}(x)X = T(x)XT(x)$, defines a C_0-semigroup on $\mathcal{K}(F)$, whose generator $\Phi_{A,A}$ is formally given by

$$\Phi_{A,A}X = \lim_{x \to 0} \frac{\mathcal{T}(x)X - X}{x} = \frac{d}{dx}\mathcal{T}(x)X\Big|_{x=0} = \frac{d}{dx}\Big(T(x)XT(x)\Big)\Big|_{x=0} = AX + XA.$$

Obviously, $\Phi_{A,A}$ is only defined for those X whose image lies in $D(A)$. Criteria for the invertibility of $\Phi_{A,A}$ have been gathered in Proposition 2.15.

We summarize:

Proposition 4.4 *(analogous to [12] Proposition 3.6) Let F be a Banach space with metric approximation property. Let A and A^3 be generators of C_0-semigroups $(e^{xA})_{x\geq 0}$, $(e^{tA^3})_{t\geq 0}$ on F, and let $\Phi_{A,A}$ be invertible.*

Furthermore suppose $B \in \mathcal{N}(F)$ to be an operator whose image lies in $D(A^k)$, A^kB to be nuclear, and the equation $ABf + BAf = (a \otimes c)f$ to be valid $\forall f \in D(A)$ with $c \in D(A^k)$ $(1 \leq k \leq 4)$.

Then $u = 2\partial_x^2 \log \det_{\mathcal{N}}(1 + e^{xA}e^{tA^3}B)$ solves the KdV (1) whenever the determinant is non-vanishing.

A very natural device to guarantee the invertibility of $\Phi_{A,A}$ consists in requiring for $\omega(A) = \inf\{w \in \mathbb{R} |$ there is $M \geq 0$ such that $\|T(x)\| \leq Me^{wx}$ for all $x \geq 0\}$, the so-called growth estimate, $\omega(A) < 0$. Then assumption (i) in Proposition 2.15 is fulfilled. More precisely, the following properties of the C_0-semigroup $(\mathcal{T}(x))_{x\geq 0}$ on $\mathcal{N}(F)$,

$$\int_0^{x_0} \mathcal{T}(x)(a \otimes c)dx \in D(\Phi_{A,A}) \quad \text{and} \quad \Phi_{A,A} \int_0^{x_0} \mathcal{T}(x)(a \otimes c)dx = \mathcal{T}(x_0)(a \otimes c) - (a \otimes c),$$

yield in the limit $x_0 \to \infty$:

Lemma 4.5 *Let $-A \in L(F)$ be the generator of a C_0-semigroup $(T(x))_{x\geq 0}$ on F with $\omega(-A) < 0$. Then the operator B defined by the Bochner integral $B = \int_0^\infty T(x)(a \otimes c)T(x)dx$ exists and is nuclear, and we have $B \in D(\Phi_{A,A})$ and $AB + BA = a \otimes c$.*

To illustrate this extension of our operator method to unbounded operators, we once more turn to the construction of solutions describing countable superpositions of solitons (see Subsection 3.2).

First we shall explain how one can get new solutions of this kind using Proposition 4.4 and Lemma 4.5.

Let F be one of the classical sequence spaces c_0 or (a weighted) ℓ_p, $1 \leq p < \infty$, and A a diagonal operator on F defined by a (not necessarily bounded) sequence $(k_i)_i$, i.e.

$$A : F \to F \quad \text{with} \quad A(\xi_i)_i = (k_i \xi_i)_i.$$

Multiplication operators provide standard examples in the theory of C_0-semigroups (confer for example [40] Section A-I.2), and the following is well-known:

$$-A \text{ generates a } C_0\text{-semigroup } (T(x))_{x\geq 0} \iff \sup_i \Re(-k_i) < \infty.$$

Moreover, $\omega(-A) = \sup_i \Re(-k_i)$.

To guarantee the existence of $B := \Phi_{A,A}^{-1}(a \otimes c)$, by Lemma 4.5 it is thus sufficient to assume

$$\inf_i \Re(k_i) > 0. \tag{21}$$

Finally, if $c \in D(A^k)$, $1 \leq k \leq 4$, then we are in position to apply Proposition 4.4. Obviously, this way we obtain solutions u on $([0, \infty) \times [0, \infty)) \cap D(u)$.

Remark 4.6 *The sole assumption (21) on the sequence* $(k_i)_i$ *admits real sequences without upper bound. Hence the described method allows the construction of solutions which can be interpreted as countable superposition of solitons whose velocity and size gets arbitrarily big (A soliton associated to the eigenvalue* k_i *has velocity* $-k_i^2$ *and amplitude* $k_i^2/2$*).*

In addition we shall point out how to weaken the assumption on the sequence $(k_i)_i$ even more by a construction according to Proposition 4.2, Lemma 4.3 (if $(k_i)_i \subset \mathbb{R}$ set $\lambda = \mathrm{i}$).

Corollary 4.7 *(confer [12] Corollary 4.4) Let* $(k_i)_i$ *be a sequence of real numbers with* $k_i+k_j \neq 0$ $\forall i,j$ *and* $(d_i)_i$ *a sequence of positive numbers with* $(d_i)_i \in \ell_\delta$ *for some* $\delta \in (0,1)$ *so that the following properties are satisfied for* $t \in (-\infty, 0]$*:*

$$\sup_{i,j}\ (d_i d_j)^{(1-\delta)/2} \frac{\sqrt{|k_i k_j|}}{|k_i+k_j|}\ |k_i^n l_i| < \infty \ \text{and}\ \left(d_i^{(1-\delta)/2}\sqrt{|k_i|}\ (k_i^n l_i)\right)_i \in c_0 \quad \forall x \tag{22}$$

$(0 \leq n \leq 4)$, where $l_i(x,t) = \exp\left(k_i x + k_i^3 t\right)$*.*

Then $a = \left(d_i^{(1-\delta)/2}\sqrt{|2k_i|}\,\right)_i \in c_0'$*,* $c = \left(d_i^{(1-\delta)/2}\sqrt{|2k_i|}\,\mathrm{sign}(k_i)\right)_i \in c_0$ *holds true and*

$$M(x,t) = \Big((a_i)_i \otimes (c_i\, l_i)_i\Big), \qquad L(x,t) = \Big(\frac{a_j c_i}{k_i+k_j}\, d_j^\delta\, l_i\Big)_{i,j} \in \mathcal{N}(c_0)$$

fulfill the assumptions of Proposition 4.2, Lemma 4.3.

Therefore, $u = 2\partial_x^2 \log \det_{\mathcal{N}}(1+L)$ *solves the KdV (1) whenever the determinant is non-vanishing.*

Remark 4.8 *For positive sequences* $k_i > 0\ \forall i$*, which are bounded away from zero,* $\inf_i k_i > 0$ *(i.e. one considers superpositions of solitons with a certain minimal velocity), assumption (22) can be replaced by the 'physically reasonable' condition*

$$\lim_{i\to\infty} d_i^{1/k_i} = 0. \tag{23}$$

The heuristic meaning of (23) is that the single solitons are 'far away' from each other at $t=0$ *(note that the maximum of the soliton corresponding to the eigenvalue* k_i *is located in* $x_i = -(\log d_i)/k_i$ *at* $t=0$*). Such solutions are certainly defined on* $\mathbb{R}\times(-\infty,0]$*.*

In the paper [12] the authors discuss the difficulties to choose the operator A in such a manner that A and A^3 are both generators of C_0-semigroups. To this end they employ the calculus of fractional powers for m-accretive operators (operators T for which there is a $\lambda \geq 0$ such that $((-T)-\lambda I)$ has a bounded inverse (defined on the whole space)). By the Lumer-Phillips-Theorem we have:

$$T \text{ densely defined, m-accretive} \implies -T \text{ generates a } C_0\text{-semigroup of contractions.}$$

For m-accretive operators T fractional powers T^α, $\alpha \in (0,1)$, can be defined (see [59] for the details) and are again m-accretive. Moreover, the growth estimate can be calculated using $\omega(-T^\alpha) = \sup\{\Re(\lambda) | \lambda \in \mathrm{spec}(-T^\alpha)\}$ with $\mathrm{spec}(T^\alpha) = \mathrm{spec}(T)^\alpha$. Thus the invertibility of Φ_{T^α,T^α} can be reduced to Lemma 4.5: A reasonable choice is $A = -T^{1/3}$ for a given m-accretive operator T with $\omega(-T) < 0$.

This leads to the construction of further solutions in [12], whose properties differ considerably from what we examined before. For example the nilpotent C_0-semigroup of translations on $L_2(0,\tau)$ ($\tau > 0$ fixed) defined by

$$T(t)f(s) = \begin{cases} f(s+t), & s+t \leq \tau \\ 0, & s+t > \tau \end{cases}$$

yields solutions $u(x,t)$ in $[0,\infty)\times[0,\infty)$ which own the property that $u(\cdot,t)$ tends to zero faster then any exponential for any given $t \geq 0$ and that the solution itself vanishes after the predicted time τ.

4.2 The initial value problem

As already mentioned, the initial value problem

$$u_t = u_{xxx} + 6uu_x, \qquad u(x,0) = u_0(x), \tag{24}$$

can be solved for rapidly decreasing potentials $u_0(x)$ (that is $u_0 \in \mathcal{S}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ the Schwartz space) by means of the inverse scattering method.

For the KdV the (direct) scattering problem mainly consists in the investigation of the spectral properties of the Schrödinger operator H in $L_2(\mathbb{R})$,

$$H = \frac{d^2}{dx^2} + u_0(x) \quad \text{with } D(H) = \left\{ f \in L_2(\mathbb{R}) \Big| \int_{\mathbb{R}} |\hat{f}(s)|(1+|s|)^2 ds < \infty \right\} \tag{25}$$

($D(H)$ the Sobolev space $H^2(\mathbb{R})$ of second order). Corresponding to H one constructs the so-called scattering data $\Sigma(H) = \{\kappa_1, \ldots, \kappa_N; d_1, \ldots, d_N; \rho(\cdot)\}$, where

a) $\kappa_1, \ldots, \kappa_N$ are N positive numbers such that $\{\kappa_1^2, \ldots, \kappa_N^2\}$ is the discrete spectrum of H,
b) $d_1, \ldots, d_N$ are N positive normalizing constants, and
c) the reflection coefficient ρ is a continuous function defined on $\mathbb{R}\backslash\{0\}$ and fulfilling the conditions $\rho(-s) = \overline{\rho(s)}$ and $|\rho(s)| < 1$ for $s \neq 0$.

Conversely, the inverse scattering problem consists in reconstructing the potential corresponding to H from given scattering data $\Sigma(H)$. To this end one defines the kernel

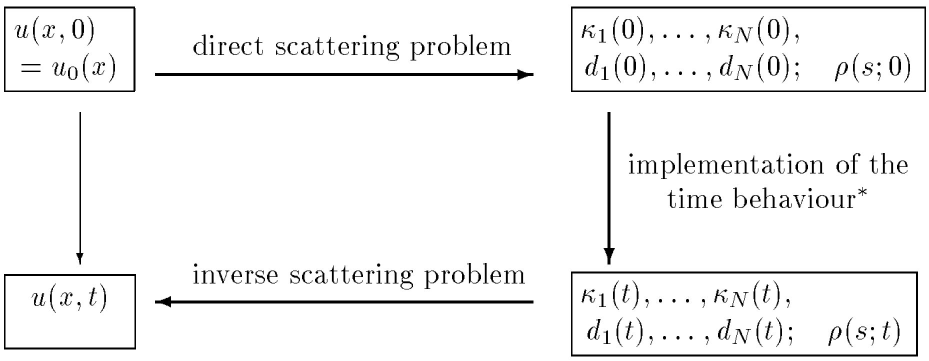
$$\Gamma(x) = \sum_{j=1}^{N} d_j e^{\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(s) e^{-\mathrm{i}sx} ds \quad \text{for } x \in \mathbb{R} \tag{26}$$

and studies the so-called Gelfand-Levitan-Marchenko equation

$$0 = \Gamma(x+y) + k(x,y) + \int_{-\infty}^{x} k(x,z)\Gamma(z+y)dz \quad (y \leq x) \tag{27}$$

for $k(x,y)$ ($y \leq x$). There is a unique solution of this equation, and it can be shown that the desired potential u_0 is obtained from the solution of the Gelfand-Levitan-Marchenko equation by setting $u_0(x) = -2\partial_x k(x,x)$. For a careful elaboration of the details we refer the reader to [10], Chapter 2.1.

The inverse scattering transform combines both, direct and inverse scattering problem, in order to solve the initial value problem of the KdV (24) as sketched in the diagram:



* The strategy for the implementation of the time behaviour is the following: for a given solution $u(x,t)$ of the initial value problem, one employs the fact that $u(x,t)$ solves the KdV to derive a formula for the time dependence of the scattering data. One observes: $\kappa_j(t) = \kappa_j(0)$, $d_j(t) = d_j(0)e^{8\kappa_j^3 t}$, and $\rho(s;t) = \rho(s;0)e^{8is^3 t}$.

Remark 4.9 *The assumption that u_0 is a rapidly decreasing potential is quite convenient to carry out the inverse scattering transform. However there are several known possibilities to weaken this assumption on u_0.*

Here we mention only results of Marchenko (confer as well Subsection 2.5) who in [31] succeeded in solving the initial value problem for very general potentials, which are no longer rapidly decreasing or periodic.

In his thesis [10] (the idea is originally due to [2]), Blohm associates operators $K(x,y)$ to given scattering data in such a way that $k(x,y) = \mathrm{tr}(K(x,y))$ solves the Gelfand-Levitan-Marchenko equation (27). Starting from the abstract result

Proposition 4.10 *([9] Theorem 3.2) Let F be a Banach space and let A generate a C_0-group $(e^{xA})_{x\in\mathbb{R}}$ on F. In addition, let $a \in F'$, $c \in F$, and $B \in \mathcal{L}(F)$ such that*

$$\langle B\hat{c}, \hat{a}\rangle = \int_0^\infty \langle \exp(-sA)\hat{c}, a\rangle \langle \exp(-sA)c, \hat{a}\rangle ds \quad \forall \hat{a} \in F', \hat{c} \in F \text{ holds,}$$

furthermore assume that the inverse in (28) exists.

Then $k(x,y) = \mathrm{tr}(K(x,y))$,

$$K(x,y) = -\Big(\exp\left((y-x)A\right)(1 + \exp(2xA)B)^{-1}\exp(2xA)\Big)(a \otimes c), \quad y \le x, \tag{28}$$

is a solution of the Gelfand-Levitan-Marchenko equation (27) with respect to the kernel $\Gamma(x) = \langle \exp(xA)c, a\rangle$.

he explicitely constructs an operator A such that the kernel $\Gamma(x)$ is adapted to the scattering data given by (26). For that, the kernel $\Gamma(x) =: \Gamma^d(x) + \Gamma^c(x)$ is decomposed in its discrete and its continuous part, and each part is treated separately. Next we shall explain the choices that lead to $\langle \exp(xA)c, a\rangle = \Gamma^{d,c}(x)$ (confer Proposition 4.10).

a) Let $A \in \mathcal{L}(\ell_2^N)$ be the diagonal operator given by $(\kappa_1, \dots, \kappa_N)$ and let the vectors a, c satisfy $a_j c_j = d_j$ $\forall j$. Then $\langle \exp(xA)c, a\rangle = \Gamma^d(x)$ holds true, and, as for the N-solitons, $B = \left(\dfrac{a_j c_i}{\kappa_i + \kappa_j}\right)_{i,j=1}^N$.

b) Let $A \in L(L_2(\mathbb{R}))$ be defined by $(Af)(s) = -\mathrm{i}sf(s)$, thus generating the C_0-group $(T(x))_{x\in\mathbb{R}}$ on $L_2(\mathbb{R})$ where $(T(x)f)(s) = e^{-\mathrm{i}xs}f(s)$. Let $a, c \in L_\infty(\mathbb{R}) \cap \bigcap_n D(A^n)$ satisfy $2\pi a(s)c(s) = \rho(s)$. Then $\langle \exp(xA)c, a\rangle = \Gamma^c(x)$ holds true. Setting $B = 2\pi M_c \mathcal{F}^{-1} I_+ P_+ \mathcal{F}^{-1} M_{\overline{a}}$, where $I_+ : L_2(0,\infty) \hookrightarrow L_2(\mathbb{R})$ and $P_+ : L_2(\mathbb{R}) \to L_2(0,\infty)$ embedding and projection, $\mathcal{F}$ the Fourier transform on $L_2(\mathbb{R})$ and M multiplication operators, all requirements of the above proposition are met.

It remains to show that, in the general case for $\Gamma(x)$, everything fits together. For the details we refer to [10].

Therefore, the initial value problem can be solved in the following manner:

For a given initial value $u(x,0) = u_0(x)$, one first uses the standard procedure to determine the scattering data and to this data one associates the kernel $\Gamma(x)$ by (26). Then the above construction yields an operator $A \in L(F)$ as well as $a \in F'$, $c \in F$ (F appropriately chosen), such that the solution $k(x,y)$ of the Gelfand-Levitan-Marchenko equation is explicitely given by Proposition 4.10. In particular $u_0(x) = -2\partial_x k(x,x)$.

On the other hand, within our formalism we obtain a solution of the KdV starting from data $2A \in L(F)$ and $\sqrt{2}a \in F'$, $\sqrt{2}c \in F$ by

$$u(x,t) = 2\partial_x \mathrm{tr}\Big(\Big((1 + \exp(2xA + 8tA^3)B\big)^{-1} \exp(2xA + 8tA^3)\Big)(a \otimes c)\Big).$$

Comparison of the formulas shows $u(x,0) = -2\partial_x k(x,x)$, thus $u(x,t)$ is a solution of the KdV with initial value $u(x,0) = u_0(x)$.

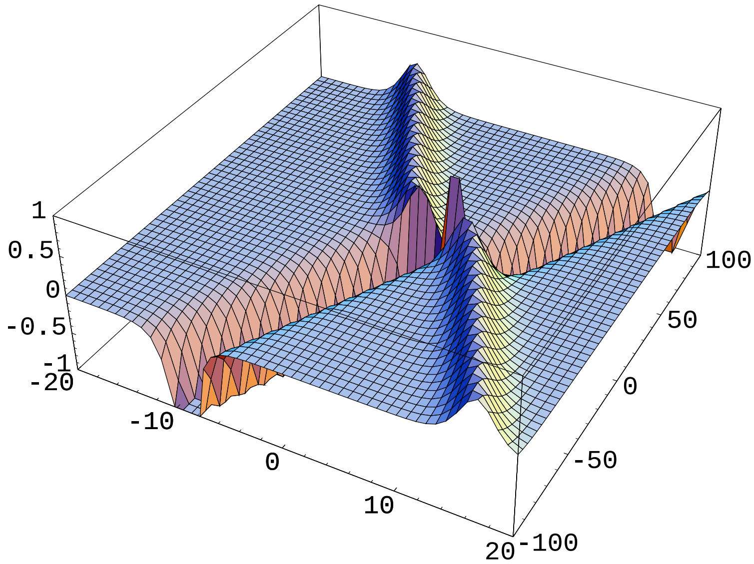
In summary, we have shown that all solutions obtained by the inverse scattering transform are also available within our formalism.

Remark 4.11 *Generating the N-solitons u from the operator* $\mathrm{diag}\{k_1,\ldots,k_N\}$*, the discrete eigenvalues of $H = d^2/dx^2 + u$ are given by $\{k_1^2/4, \ldots k_N^2/4\}$, a relation which we have already found in connection with the results of Gesztesy et al. in Subsection 3.2.*

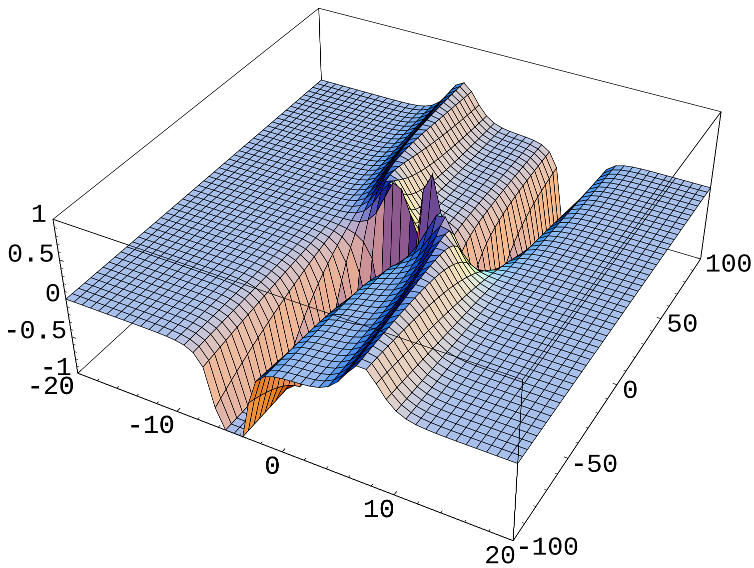
Supplement A

Subsequently, we assemble some computer graphics to illustrate our result on negatons, see Theorem 3.7. The main point is of course the difference between straight lines of solitary waves and logarithmic rays of the members of negatons. Furthermore, the diagrams underline that the convergence we established is in fact very rapid.

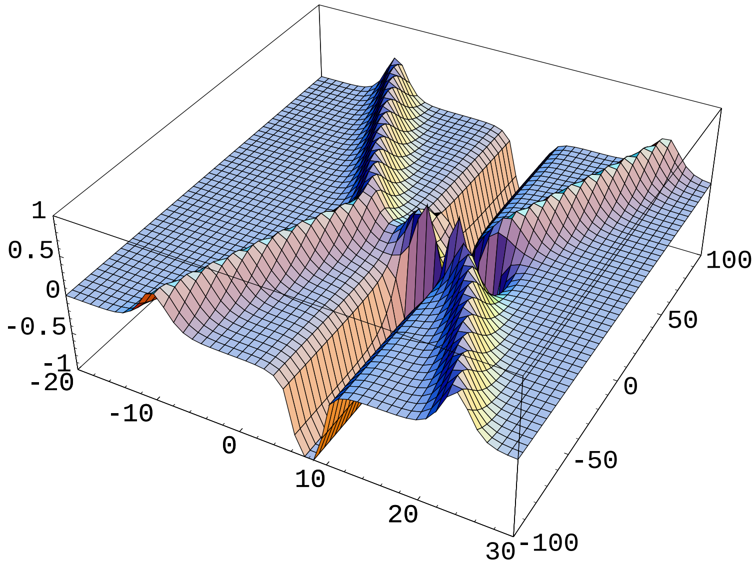
In the pictures, the solution $u(x-t,t)$ is plotted, where the variables x and t are depicted as usually.



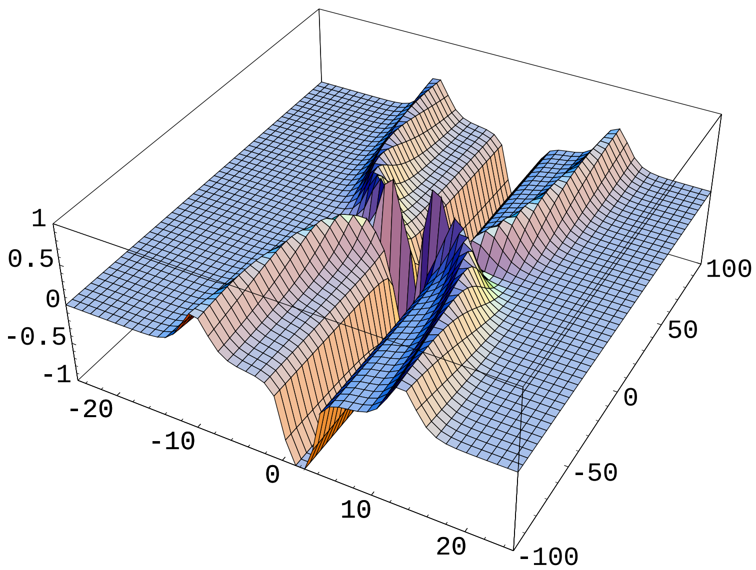
A soliton ($k_1 = 1.05$) and an antisoliton ($k_2 = 0.95$)



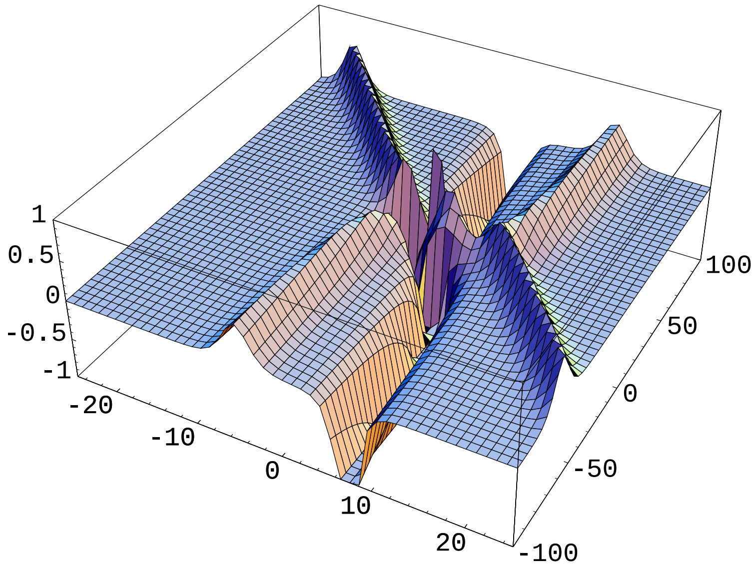
negaton ($k = 1$) consisting of a soliton and an antisoliton



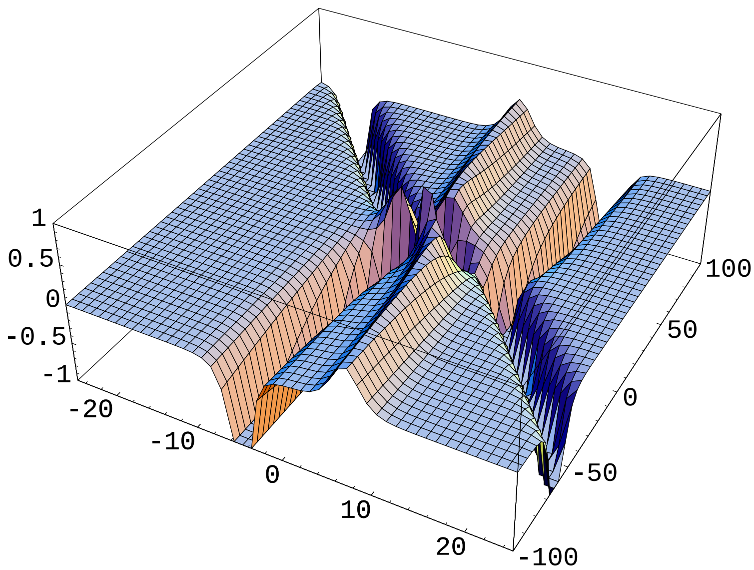
Two solitons($k_1 = 0.95$, $k_2 = 1.05$) and an antisoliton ($k_3 = 1$)



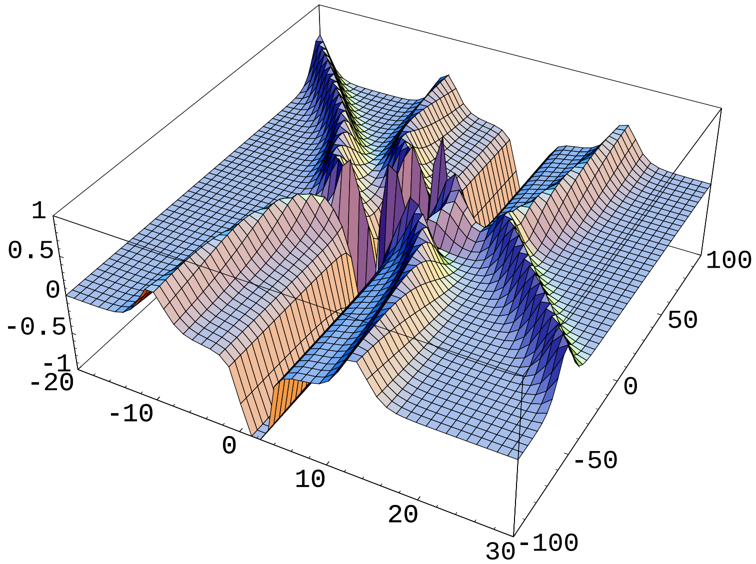
negaton ($k = 1$) consisting of solitons and an antisoliton



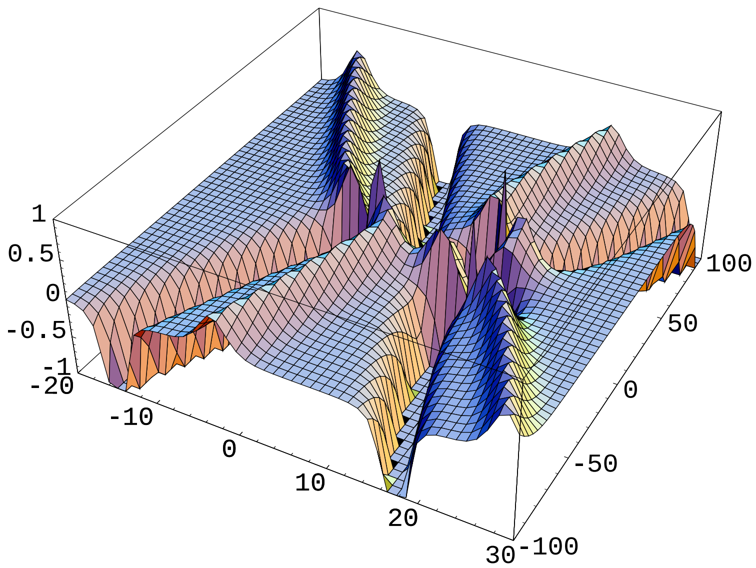
negaton ($k_1 = 1$) consisiting of an soliton and an antisoliton
as well as a soliton ($k_2 = 1.1$)



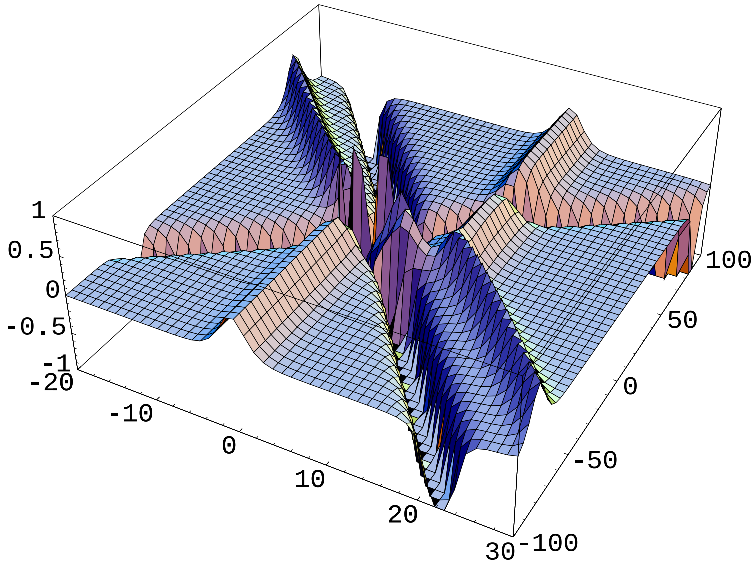
negaton ($k_1 = 1$) consisting of a soliton and an antisoliton
as well as an antisoliton ($k_2 = 1.1$)



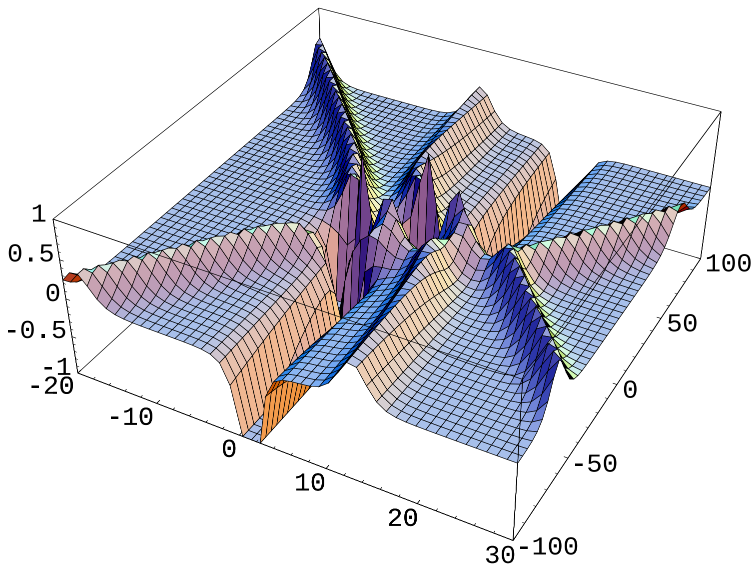
negaton ($k_1 = 1$) consisting of two solitons and an antisoliton
as well as a soliton ($k_2 = 1.1$)



negaton ($k_1 = 0.95$) consisting of a soliton und an antisoliton
as well as negaton ($k_2 = 1.05$) consisting of a soliton und an antisoliton



negaton ($k_1 = 1.1$) consisting of a soliton und an antisoliton
as well as a soliton ($k_2 = 1$) and an antisoliton ($k_3 = 0.8$)



negaton ($k_1 = 1$) consisting of a soliton and an antisoliton
as well as two solitons ($k_2 = 0.9$, $k_3 = 1.1$)

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