Real Submanifolds in Complex Space - Examples and Exercises

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Despite a certain number of monographs on CR geometry, see [2, 5, 10], there is no textbook offering problems for graduate students entering the topic. The following compendium tries to fill this gap for the needs of the Miun PhD course on real submanifolds in complex space.

1 Levi flat CR manifolds

1.1 Elementary analysis

Let $M \subset \mathbb{C}^n$ be a smooth generic Levi flat CR manifold.

a) Show that a continuous function f on M is a CR function iff f is holomorphic along the CR leaves (This becomes very easy with the Baouendi-Treves approximation theorem. It is instructive to prove this by a direct verification of the weak tangential CR equations).

b) Deduce that two continuous CR functions which coincide near p_0 coincide in a neighborhood of the CR orbit through p_0.

c) Let $N \subset M$ be a smoothly embedded totally real manifold of real dimension n. Show that a CR function whose restriction to N vanishes vanishes in a neighborhood of N in M.

Hint: Consider first the case that M is a domain in $\mathbb{C}^n$.

d) For $p_0 \in M$. It is easy to find an open neighborhood $U \subset M$ such that there is no ambiently open V with $\emptyset \neq V \cap M \subset U$ such that every continuous CR function on U admits a holomorphic extension to U. Prove also an analogous statement for extension to one-sided neighborhoods.

Remark: The parts (b) and (c) hold on every CR manifold. The proofs become much more involved.

1.2 Levi flat hypersurface with a worm cf. [7], Section 7.

In $\mathbb{C}^* \times \mathbb{C}$ consider $M = \{w \exp(i \log z) = 0\} = \{w \exp(i \log |z|) = 0\}$.

a) Show that M is Levi flat and that $(\mathbb{C}^* \times \mathbb{C}) \setminus M$ is the union of two pseudoconvex domains $D^{\pm}$.

b) Show that the closed annuli

$$R_s = \{(z, is) : \exp(-\pi/2) \leq |z| \leq \exp(-\pi/2)\}, s \in \mathbb{R},$$

all have boundary in M, but that only R_0 is contained in M.

c) Fix $a, b > 0$. Derive that any function holomorphic near $R_0 \cup \bigcup_{-a \leq s \leq b} \partial R_s$ has a holomorphic extension to a neighborhood of $\bigcup_{-a \leq s \leq b} R_s$.

Remark: The last statement shows that the domains $D^{\pm}$ have nontrivial Nebenhülle, i.e. that there is no pseudoconvex neighborhood basis of the closures. It is even possible to construct pseudoconvex smoothly bounded domains $D \Subset \mathbb{C}^n$ with similar properties, see [6], Lectures 24, 25.

1.3 Global geometry

a) Show that $\mathbb{C}^n$ does not contain an embedded closed Levi flat CR manifold of positive CR dimension.

Hint: Let R be the minimal positive number such that $M \subset \overline{B_r(0)}$ (Why does R exist?). Show that $M \cap S_R(0) \neq \emptyset$ and that M is not Levi flat in the points of $M \cap S_R(0)$.

b) Find a complex manifold containing an embedded closed Levi flat CR manifold of positive CR dimension.

Remark: It is still an open problem whether $\mathbb{CP}^2$ contains an embedded closed Levi flat hypersurface.

2 Brackets and Levi form

2.1 Distributions in the tangent bundle.

Let M be a smooth real manifold of dimension n and K a smooth rank-k subbundle of TM.

a) Define inductively $\mathcal{G}_1 = \Gamma^\infty(M, K)$ and for $j \geq 1$

$$\mathcal{G}_{j+1} = \mathcal{G}_j + [\mathcal{G}_1, \mathcal{G}_j].$$

For $p_0 \in M$ fixed define $G_j = G_{j,p_0} = \{X(p_0) : X \in \mathcal{G}_j\}$. Show that the mapping

$$(X_1, \ldots, X_j) \mapsto [X_1, \ldots, X_j](p_0) \bmod \mathcal{G}_{j-1}$$

only depends on the pointwise values $X_1(p_0), \ldots, X_j(p_0)$[1]. Hence one obtains a mapping from $(\mathcal{G}_1)^j$ to $\mathcal{G}_j/\mathcal{G}_{j-1}$.

b) Consider the analogous construction for complex vector fields with values in $K^c = \mathbb{C} \otimes K = K \oplus iK$. Show that $G_j = T_{p_0}M$ for sufficiently large j if and only if $G_j^c = \mathbb{C} \times T_{p_0}M$ for sufficiently large j.

c) Let M be a CR manifold and $K = T^cM$. Relate the Levi form to at least one of the above constructions.

d) Assume in (a) that there is some *ell* such that $\dim G_{\ell,p}$ does not depend on $p \in M$ and such that $G_{\ell+j,p} = G_{\ell,p}$ for all $p \in M$ and $j \in \mathbb{N}$. Prove that $\bigcup_{p\in M} G_{\ell,p}$ is a Frobenius integrable subbundle of TM.

3 Minimal points and global minimality

It is known that an arbitrary smooth CR manifold M becomes locally minimal in every point after a suitable $\mathcal{C}^k$-smooth deformation. In the following exercises we construct such deformations explicitly in a simple setting.

3.1 First properties.

Consider a smooth generic CR manifold $M \subset \mathbb{C}^n$.

a) Prove that M is minimal in $p \in M$ if and only if p possesses a neighborhood basis $\{U_j\}$ of globally minimal open subsets of M.

b) Assume that M is contained in some Levi flat hypersurface $H \subset \mathbb{C}^n$. Can M have minimal points?

3.2 Deformations in $\mathbb{C}^2$.

For $M = \{y_2 = 0\} \subset \mathbb{C}^2$ consider deformations

$$M^g = \{y_2 = g(z_1, x_2)\},\ g \in \mathcal{C}^\infty(\mathbb{C} \times \mathbb{R}).$$

[1]Read $[X_1, \ldots, X_j] = [X_1, [X_2, [\ldots, [X_{j-1}, X_j] \ldots]$

Construct a deformation g with $g(0) = 0$, $dg(0) = 0$ such that M^g is minimal in 0. Show that g can be chosen with support in a given neighborhood of 0 and of arbitrary small $\mathcal{C}^2$-norm.

Hint: One may choose g with $g = 0$ for $x_2 \leq 0$. Observe that for such g the local orbit through 0 is either open or lies in $\{z_2 = 0\}$.

3.3 Deformations in $\mathbb{C}^3$.

In an analogous way we consider $\mathbb{R}^2$-valued deformations $g(z_1, x_2, x_3)$ of $M = \{y_2 = y_3 = 0\} \subset \mathbb{C}^3$.

a) Let $U \subset \mathbb{C} \times \mathbb{R}^2$ be a neighborhood of $(\hat{z}_1, 0, 0)$. with $0 \notin \overline{U}$. Construct g with support in U such that the global CR orbit of M^g through the origin is open.

Hint: One may take $g = g_1 + g_2$ where g_j have disjoint support and where $g_1 = (h_1(z_1, x_2, x_3), 0)$ and $g_2 = (0, h_2(z_1, x_2, x_3))$. What is the image of the vector-valued Levi form for such deformations?

b) Construct a deformation g with $g(0) = 0$, $dg(0) = 0$ such that M^g is minimal in 0.

3.4 Global minimality, cf. [9], proof of Lemma 5.

Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a domain with smooth connected boundary M.

a) Prove that M has only one CR orbit following the following scheme:

- **i)** If there is an orbit $N \subset M$ of dimension $2(n-1)$, then $\overline{N}$ is a union of orbits of dimension $2(n-1)$.
- **ii)** Derive a contradiction by applying the maximum principle to the orbits contained in $\overline{N}$.
- **iii)** Deduce that M has only one CR orbit.

b) Let $K \subset \mathbb{C}^n$ be a polynomial convex compact set[2]. Generalize the above argument in order to prove that each connected component of $M \backslash K$ has only one CR orbit.

[2] K is polynomially convex if for every $p \notin K$ there is a polynomial $P(z) \in \mathbb{C}[z_1, \ldots, z_n]$ such that $|P(p)| > \max_{z \in K} |P(z)|$

4 Wedge extension

An open cone $C \subset \mathbb{C}^n$ with vertex in the origin is an open set C such that $z \in C$ implies $tz \in C$ for $t > 0$. A truncated open cone is the intersection of an open cone with an open ball with center in 0. Let U be a relative domain in a generic CR manifold. An open wedge cW with edge U is a set of the form

$$U + \mathcal{W} = \{p + z : p \in U, z \in C\},$$

where C is an open truncated cone.

4.1 Intersection of hypersurfaces.

Suppose that $H_1, H_2 \subset \mathbb{C}^n$, $n \geq 3$, are strictly pseudoconcave hypersurfaces which intersect transversally in a generic CR manifold $M = H_1 \cap H_2$.

a) If all $f \in CR(M)$ have a holomorphic extension to an open wedge $\mathcal{W}$, then $\mathcal{W}$ is contained in the intersection of the pseudoconvex sides of the H_j.

Hint: You can use Kohn's result that a pseudoconvex smoothly bounded domain $D \Subset \mathbb{C}^n$ admit holomorphic functions which are smooth up to the boundary but do not extend holomorphically past any boundary point.

b) Construct an example without extension to an *open* wedge.

Suggestion: Start with the unit sphere $S^5 \subset \mathbb{C}^3$ and a suitable deformation H of S^5 such that $H \cap S^5 = \{y_3 = 0\}$.

5 Strictly pseudoconcave hypersurfaces

5.1 Penrose hypersurface, cf. [8].

Consider the Penrose hypersurface $M = \{z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 = 1 + |z_3|^2\}$.

a) Show that M is strictly pseudoconcave in every point.

b) Show that through every point pass two complex lines contained in M.

c) Find for every point a relative domain $U \subset M$ and an ambient domain $V \subset \mathbb{C}^3$ such that $\emptyset \neq V \cap M \subset U$ and every $f \in CR(U)$ has a holomorphic extension to V.

Hint: You may use the Baouendi-Treves approximation theorem combined with rigid translations of complex lines contained in M.

d) Show that there is an ambient domain $V \subset \mathbb{C}^3$ containing M such that every $f \in CR(M)$ has a holomorphic extension to V.

e) Deduce that every $f \in CR(M)$ has a holomorphic extension to $\mathbb{C}^3$.

f) Deduce that there are no closed strictly pseudoconvex CR manifolds in $\mathbb{C}^n$.

g) Show that the closure of M in $\mathbb{CP}^3$ is a strictly pseudoconvex hypersurface.

6 Homogenous weakly pseudoconcave CR manifolds

6.1 Flag manifolds

Consider the set

$$X = \{(\ell_1, \ell_2) : \ell_j \text{ is a complex subspace of } \mathbb{C}^3 \text{ of dimension } j, \, \dim \ell_j = j\}.$$

a) Show that X can be equipped with the structure of a 3-dimensional complex manifold by using charts of the following type: For each element of

$$X_{1,2} = \{(\ell_1, \ell_2) \in X : \ell_1 \not\subset \{z_1 = 0\}, \ell_2 \not\supset \{z_1 = z_2 = 0\}\}$$

there are unique vectors $v_1 = (1, z_1, z_2)$, $v_2 = (0, 1, z_3)$ such that $\ell_1 = \mathbb{C}v_1$, $\ell_2 = \mathrm{span}_{\mathbb{C}}(v_1, v_2)$. Then (z_1, z_2, z_3) are coordinates on $X_{1,2}$.

b) The special group $SL_3(\mathbb{C})$ acts on $\mathbb{C}^3$ by multiplication on the left. Show that this action induces an action on X.

6.2 CR submanifolds of X, cf. [1]

Let J be the (3×3)-matrix $\mathrm{diag}(-1, 1, 1)$, inducing the hermitian form $(u, v)_{1,2} = u^t J \bar{v}$. Let $SU_{1,2}(\mathbb{C})$ be the corresponding group of special unitary matrices, i.e. (3×3)-matrices A such that $\det A = 1$ and $(u, v)_{1,2} = (Au, Av)_{1,2}$ for all $u, v \in \mathbb{C}^3$.

a) Show that the following sets are the orbits of the action of $SU_{1,2}(\mathbb{C})$ on X:

$$\begin{aligned}
M_0 &= \{(\ell_1, \ell_2) \in X : \ell_1 \subset \ell_1^\perp, \ell_2 \supset \ell_2^\perp\}, \\
M_1 &= \{(\ell_1, \ell_2) \in X : \ell_1 \subset \ell_1^\perp, (\cdot, \cdot)_{1,2}|_{\ell_2}, \text{ is nondegenerate}\}, \\
M_2 &= \{(\ell_1, \ell_2) \in X : (\cdot, \cdot)_{1,2}|_{\ell_1} < 0\}, \\
M_3 &= \{(\ell_1, \ell_2) \in X : (\cdot, \cdot)_{1,2}|_{\ell_1} > 0, \ell_2 \supset \ell_2^\perp\}, \\
M_4 &= \{(\ell_1, \ell_2) \in X : (\cdot, \cdot)_{1,2}|_{\ell_1} > 0, (\cdot, \cdot)_{1,2}|_{\ell_2}, \text{ is nondefinite}\}, \\
M_5 &= \{(\ell_1, \ell_2) \in X : (\cdot, \cdot)_{1,2}|_{\ell_1} > 0, (\cdot, \cdot)_{1,2}|_{\ell_1} > 0\}.
\end{aligned}$$

Hint: Use Witt's theorem that an isometry between subspaces can be extended to an isometry of $\mathbb{C}^3$.

b) Show that the above M_j are generic CR manifolds. Determine the CR dimension in each case.

c) Show that M_1 is weakly but not strictly pseudoconcave.

Hint: Use the projection $(\ell_1, \ell_2) \mapsto \ell_1 \in \mathbb{P}^2$. Observe that it maps M_1 to a manifold isomorphic to $S^3 \subset \mathbb{C}^2$.

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