Mittuniversitetet
MID SWEDEN UNIVERSITY

# Counting Statistics and Error Prediction

# Outline

- Characterization of data
- Statistical models
- Error propagation
- Optimization of counting experiments
- Energy resolution
- Detection efficiency
- Dead time

# Characterization of data

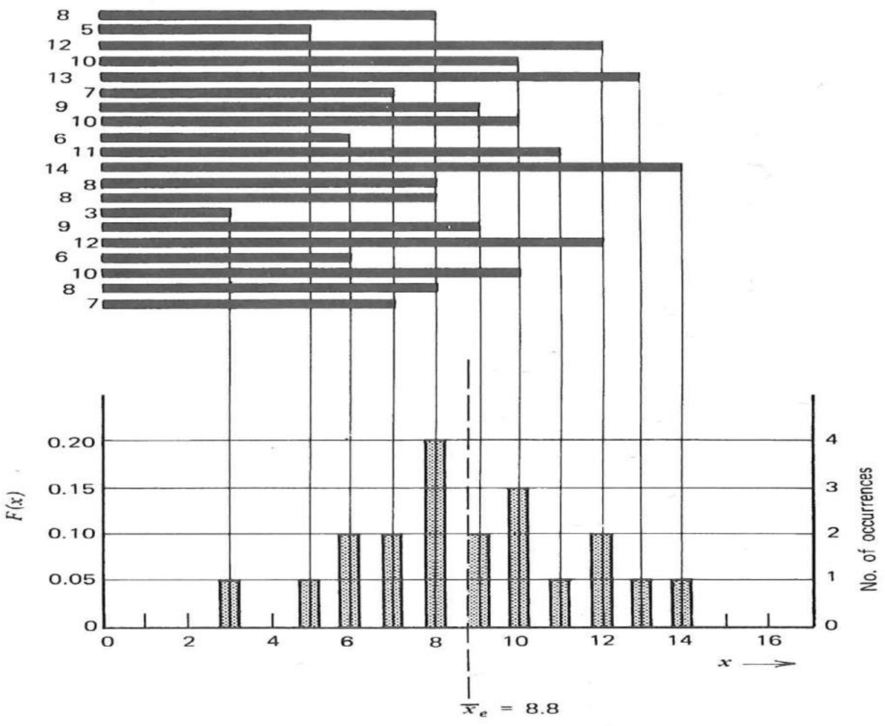
- *N* independent measurements
  - $X_1..X_2..X_3..X_4.......X_N$
- Sum $\sum \equiv \sum_{i=1}^{N} x_i$
- Experimental mean $\bar{x} \equiv \sum / _N$
- Frequency distribution function $F(x) \equiv \dfrac{number\ of\ the\ occurences\ of\ the\ value\ x}{number\ of\ measurements\ (=N)}$
- $\sum_{x=0}^{\infty} F(x) = 1$
- $\overline{x} = \sum_{x=0}^{\infty} xF(x)$ (by using Frequency distribution function)

Mittuniversitetet
MID SWEDEN UNIVERSITY

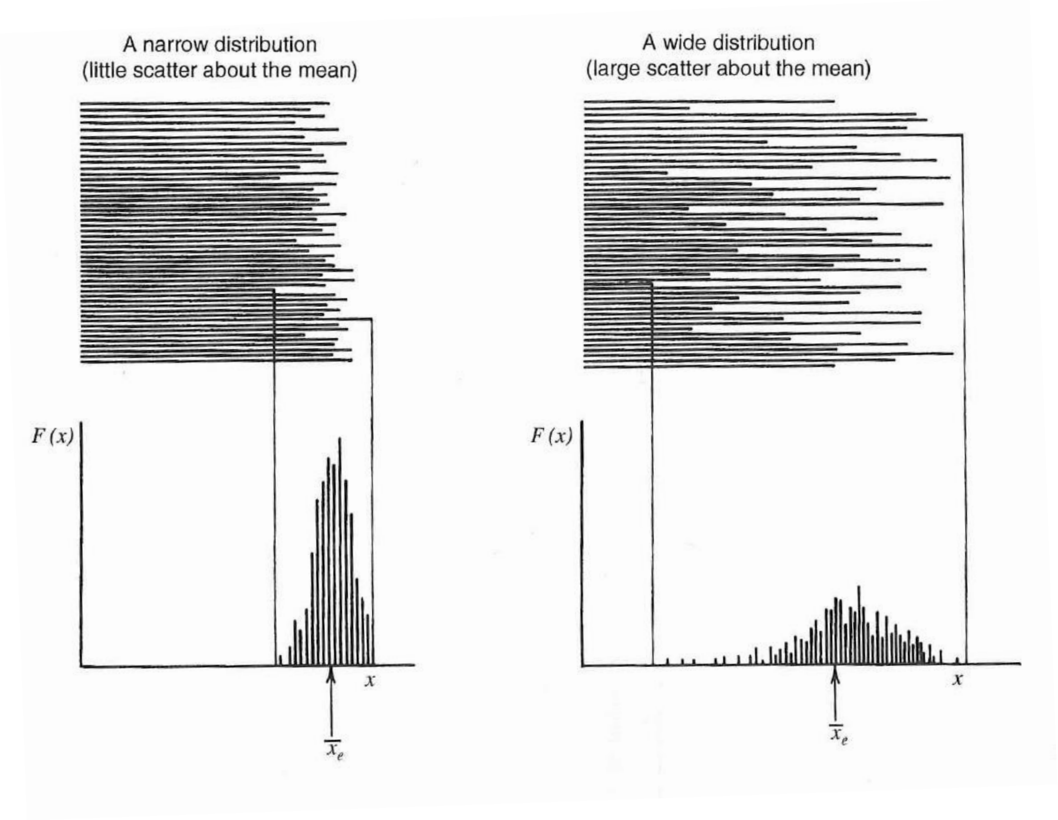
# Characterization of data

**Table 3.1** Example of Data Distribution Function

| Data | | Frequency Distribution Function | |
|---|---|---|---|
| 8 | 14 | $F(3) = 1/20$ | $= 0.05$ |
| 5 | 8 | $F(4)$ | $= 0.00$ |
| 12 | 8 | $F(5)$ | $= 0.05$ |
| 10 | 3 | $F(6)$ | $= 0.10$ |
| 13 | 9 | $F(7)$ | $= 0.10$ |
| 7 | 12 | $F(8)$ | $= 0.20$ |
| 9 | 6 | $F(9)$ | $= 0.10$ |
| 10 | 10 | $F(10)$ | $= 0.15$ |
| 6 | 8 | $F(11)$ | $= 0.05$ |
| 11 | 7 | $F(12)$ | $= 0.10$ |
| | | $F(13)$ | $= 0.05$ |
| | | $F(14)$ | $= 0.05$ |
| | | $\sum_{x=0}^{\infty} F(x)$ | $= 1.00$ |

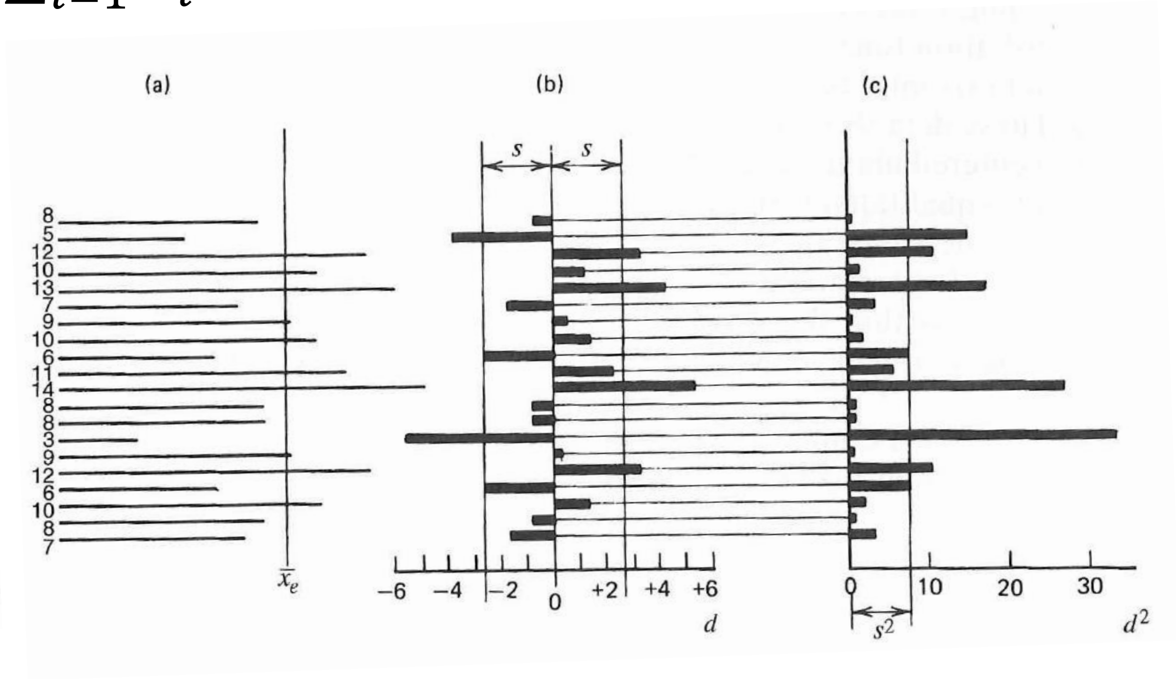


# Characterization of data




$d_i = x_i - \bar{x}$ residual

$\sum_{i=1}^{N} d_i = 0$



# Characterization of data

- $\varepsilon_i \equiv x_i - \mu$ deviation (real mean)
- $s^2 \equiv \overline{\varepsilon^2} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$ (Variance) or $s^2 = \sum_{x=0}^{\infty}(x - \mu)^2\,\mathrm{F(x)}$ or
  $s^2 = \overline{x^2}\text{-}\mu^2$
- $s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$ (using sample mean)


- Any set of data can be completely described by its frequency distribution function F(x)
- Particular if interested are: sample mean and sample variance

## Statistical models

- (The binomial distribution)
- The Poisson distribution “discrete distribution”

  Used where events occur randomly and independent in space or time
- The normal distribution “continuous distribution”
  - Central limit theorem (CLT)
    - Sum (mean value) of a sufficient number of independent and identically distributed random variables is approximately normally distributed

## Statistical models

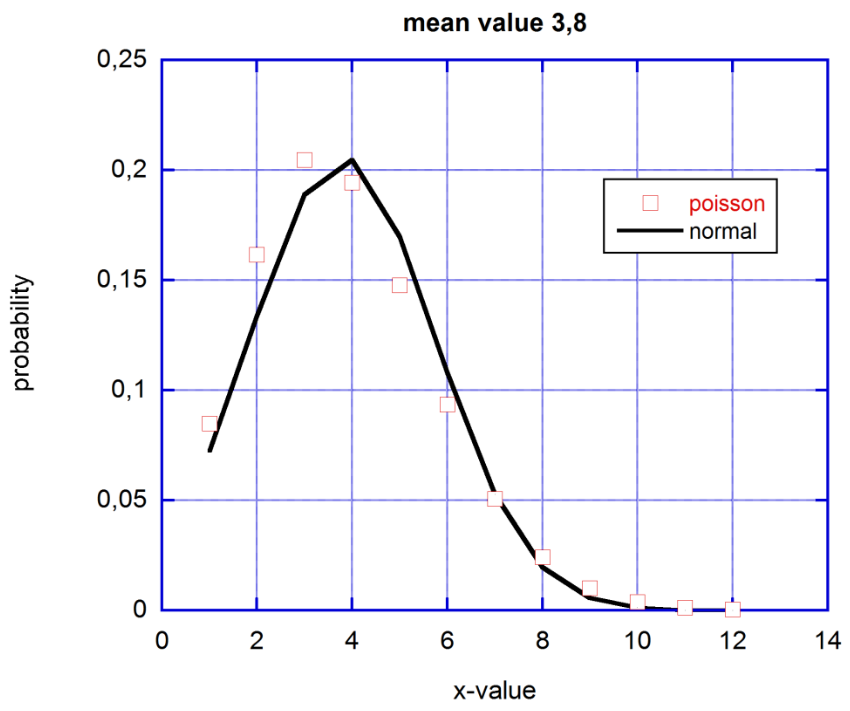
- The Poisson distribution
- $P(x) = \frac{(\mu)^x e^{-\mu}}{x!}$ $\sum_{x=0}^{n} P(x) = 1$
- Mean value $\sum_{x=0}^{n} xP(x) = \mu$
- Distribution variance $\sigma^2 \equiv \sum_{x=0}^{n}(x - \bar{x})^2 \mathrm{P(x)} = \mu$
- Distribution standard deviation $\sigma = \sqrt{\mu}$

# Statistical models

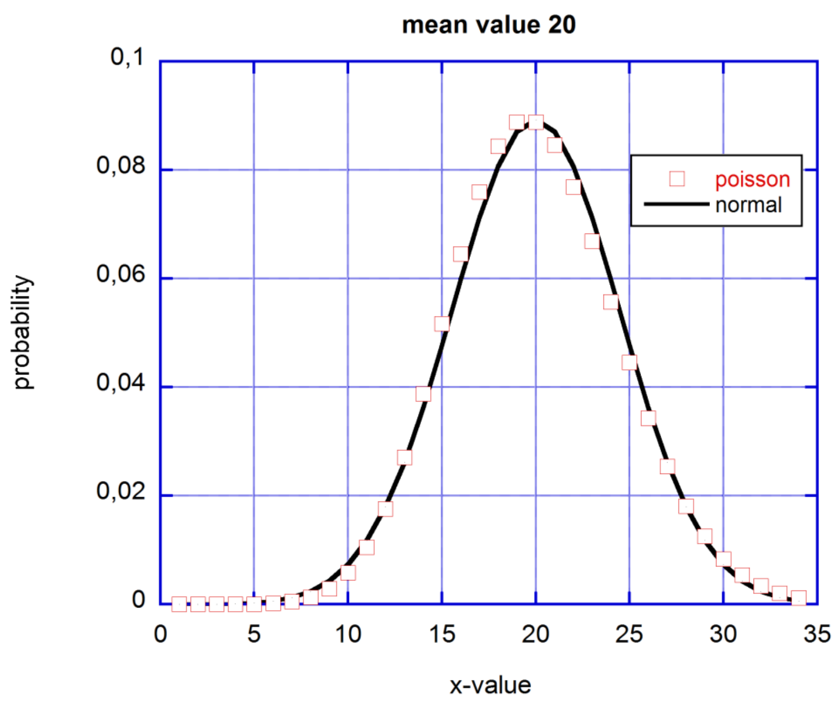
- The standard normal distribution Y∈N(0,1) (tabulated)
  - $\bar{x} = 0,\ \sigma = 1$
  - $P(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
- The normal distribution X∈N($\mu, \sigma$)
  - $\boldsymbol{P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}}$
  - Transformation from X to Y is by; $Y = \frac{X-\mu}{\sigma}$

## Statistical models

- Approximation of Poisson distribution by the normal approximation (Gaussian app.)




$$\sigma = \sqrt{\bar{x}}$$



# Statistical models

- Finding the probability in-between interval

This theorem can be used to find probabilities about $X$, which is $N(\mu, \sigma^2)$, as follows:

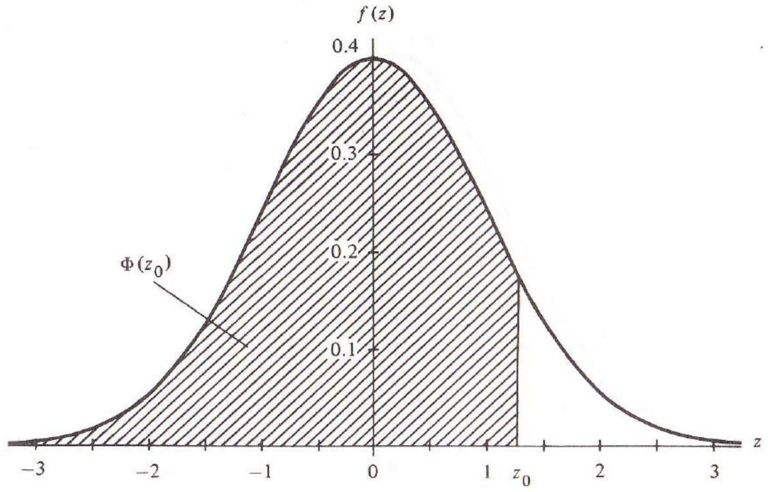
$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right),$$

since $(X-\mu)/\sigma$ is $N(0, 1)$.

## Statistical models

- Finding the probability in-between interval, example
- If X∈ $N(3{,}16)$ then $P(4 \leq X \leq 8) = P(\frac{4-3}{4} \leq \frac{X-3}{4} \leq \frac{8-3}{4}) =$

  $\emptyset(1.25)$-$\emptyset(0.25) = [tabulated] = 0.8944 - 0.5987$
  $=0.2957$




$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

## Statistical models

- Confidence interval for means with known variances

$$P\left(-z_{\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = 1-\alpha.$$

$$P\left[\bar{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \leq \mu \leq \bar{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right] = 1-\alpha.$$

The interval is given by:

$$\left[\bar{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \bar{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

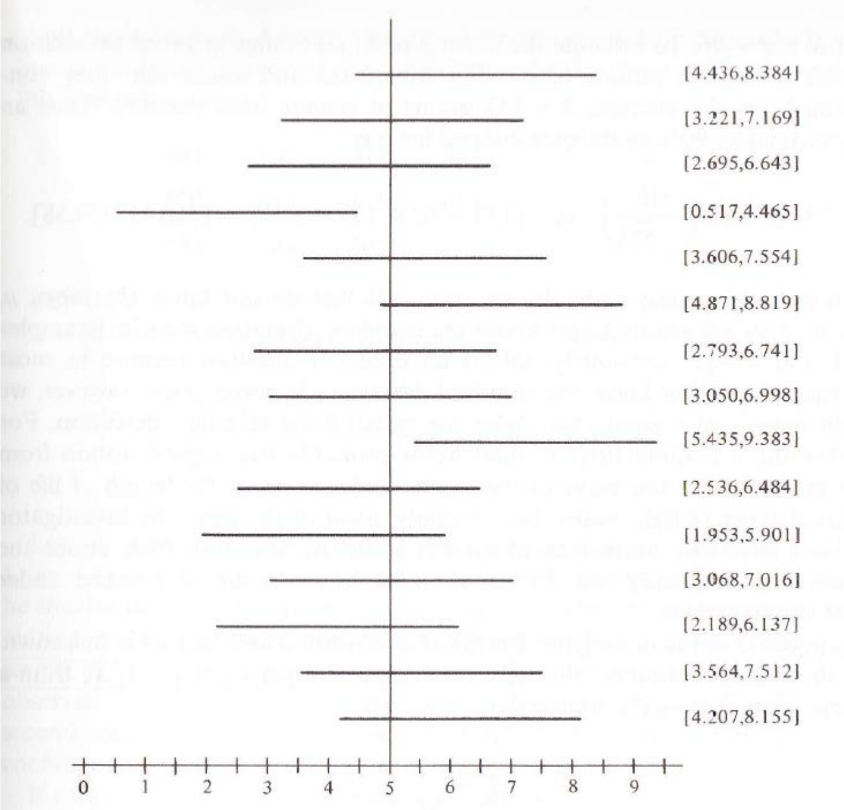
The number of probability is given by 1- $\alpha$
$z_{\alpha/2}$ is tabulated

## Statistical models

- Confidence interval for means with known variances

**Example 6.4-2** Let $\bar{x}$ be the observed sample mean of 16 items of a random sample from the normal distribution $N(\mu, 23.04)$. A 90% confidence interval for the unknown mean $\mu$ is

$$\left[\bar{x} - 1.645\sqrt{\frac{23.04}{16}},\ \bar{x} + 1.645\sqrt{\frac{23.04}{16}}\right].$$




90% confidence interval result in an $\alpha = 0.1$ $(0.9 = 1 - \alpha)$ and $z_{\alpha/2} = 1.646$

Let μ=5 computer simulation of the interval gives:

13 of 15 simulated interval contain μ=5 (86%)

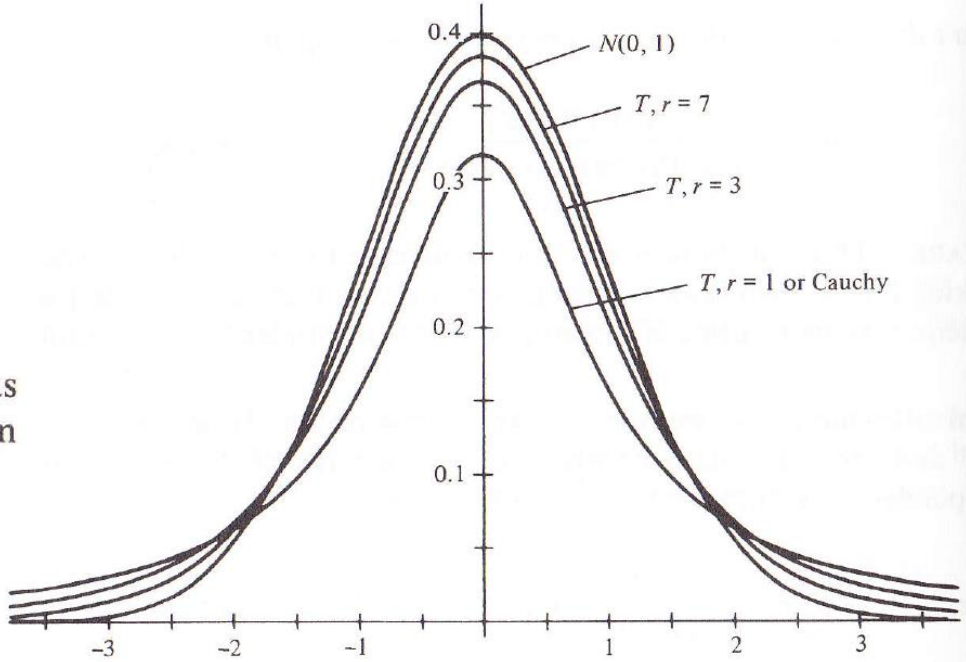
## Statistical models

- Confidence interval for means with estimated variances, t distribution

r is the degree of freedom r=n-1, large number of r result in normal distribution

$$g(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)(1+t^2/r)^{(r+1)/2}}, \qquad -\infty < t < \infty.$$

REMARK This distribution was first discovered by W. S. Gosset when he was working for an Irish brewery. Because Gosset published under the pseudonym Student, this distribution is sometimes known as Student's $t$ distribution.



# Statistical models

- Confidence interval for means with estimated variances, t distribution

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a $t$ distribution with $r = n - 1$ degrees of freedom, where $S^2$ is the usual unbiased estimator of $\sigma^2$. Select $t_{\alpha/2}(n-1)$ so that $P[T \geq t_{\alpha/2}(n-1)] = \alpha/2$. Then

$$1 - \alpha = P\left[-t_{\alpha/2}(n-1) \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{\alpha/2}(n-1)\right]$$

$$= P\left[\bar{X} - t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}}\right].$$

Thus the observations of a random sample provide $\bar{x}$ and $s^2$ and

$$\left[\bar{x} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}},\ \bar{x} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right]$$

is a $100(1-\alpha)\%$ confidence interval for $\mu$.

n=20 assume that the population is normal distributed

| | | | | | | |
|---|---|---|---|---|---|---|
| 481 | 537 | 513 | 583 | 453 | 510 | 570 |
| 500 | 457 | 555 | 618 | 327 | 350 | 643 |
| 499 | 421 | 505 | 637 | 599 | 392 | |

For these data, $\bar{x} = 507.50$ and $s = 89.75$. Thus a point estimate of $\mu$ is $\bar{x} = 507.50$. Since $t_{0.05}(19) = 1.729$, a 90% confidence interval for $\mu$ is

$$507.50 \pm 1.729\left(\frac{89.75}{\sqrt{20}}\right),$$

$$507.50 \pm 34.70, \text{ or equivalently, } [472.80, 542.20].$$

if n is larger than 30, nonnormal distribution can be approximated with normal distribution and by t student distribution, Central limit theorem (CLT)

# Error propagation

- The error propagation formula: $\sigma_f\ ^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x\ ^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y\ ^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_z\ ^2 \ldots\ldots\ldots\ldots,$ f(x,y,z,...)
- Valid only if the variables x,y,z.... is independet

# Error propagation

- Data processed by multiplication, addition or other functional manipulation
- Addition and Subtraction: f = x + y or f = x - y
  - $\frac{\partial f}{\partial x} = 1 \qquad \frac{\partial f}{\partial y} = \pm 1$
  - $\sigma_f{}^2 = (1)^2 \sigma_x{}^2 + (\pm 1)^2 \sigma_y{}^2$
  - $\sigma_f = \sqrt{\sigma_x{}^2 + \sigma_y{}^2}$

## Error propagation

- Example
  - Z=x-y net counts= total count-background counts

Total counts =1071 (x)

Background counts=521 (y)

Net counts =550 (z)

$\sigma_x = \sqrt{x}$ (Distribution standard deviation $\sigma = \sqrt{\mu}$)

$\sigma_y = \sqrt{y}$

$\sigma_z = \sqrt{x + y} = \sqrt{1592} = 40$

Net counts= $550 \pm 40$

## Error propagation

- Multiplication or Division by a Constant
  - $f = Ax$
  - $\frac{\sigma f}{\sigma x} = A$
  - $\sigma_f = A\sigma_x$
  - g=x/B
  - $\sigma_g = \frac{\sigma_x}{B}$
  - Conclusion the fractional error is the same by examine the $\frac{\sigma_f}{f}, \frac{\sigma_g}{g}$ and compare with $\frac{\sigma_x}{x}$
  - Count rate$\equiv r = \frac{x}{t}$ x=1120 t=5s r=1120/5=224 $s^{-1}$ $\sigma_r = \frac{\sigma_x}{t} = \frac{\sqrt{1120}}{5} = 6.7$ r=224$\pm$6.7

# Error propagation

- Multiplication or Division of counts
- f=xy or g=$\frac{x}{y}$
- $\frac{\sigma_f}{\sigma_x} = y \ \frac{\sigma_f}{\sigma_y} = x \ \ \{the\ error\ propagation\ formula\}\ {\sigma_f}^2 = y^2{\sigma_x}^2 + x^2{\sigma_y}^2$
- Divide with $f^2=x^2y^2$
- $\left(\frac{\sigma_f}{f}\right)^2=\left(\frac{\sigma_x}{x}\right)^2+\left(\frac{\sigma_y}{y}\right)^2$ fractional error, gives the same result for division $\left(\frac{\sigma_g}{g}\right)^2=\left(\frac{\sigma_x}{x}\right)^2+\left(\frac{\sigma_y}{y}\right)^2$

## Error propagation

- Example
- count from source 1  N1=16265
- count from source 2  N2=8192
- Activity ratio R=N1/N2=1.985
- $\left(\frac{\sigma_R}{R}\right)^2=\frac{N_1}{{N_1}^2}+\frac{N_2}{{N_2}^2}=1.835\text{x}10^{-4}$   $\frac{\sigma_R}{R}=0.0135$   $\sigma_R = 0.027$   $R=1.985 \pm 0.027$

# Error propagation

- Mean value of Multiple Independent Counts

$$\Sigma = x_1 + x_2 + \cdots + x_N$$

$$\sigma_\Sigma^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots + \sigma_{x_N}^2$$

$$\sigma_\Sigma^2 = x_1 + x_2 + \ldots + x_N = \Sigma$$

$$\sigma_\Sigma = \sqrt{\Sigma}$$

$$\sigma_{\bar{x}} = \frac{\sigma_\Sigma}{N} = \frac{\sqrt{\Sigma}}{N} = \frac{\sqrt{N\bar{x}}}{N}$$

$$\bar{x} = \frac{\Sigma}{N}$$

$$\boxed{\sigma_{\bar{x}} = \sqrt{\frac{\bar{x}}{N}}}$$

The $x_i$ value can be single sample or multiple sample. If the measurement period in both cases are the same so the total number of particles are equal then the mean value and the standard deviation are equal. To improve the statistical precision by a factor 2, the number of sample (N) must be increased by a factor 4

# Error propagation

| Function | Variance | Standard Deviation |
|---|---|---|
| $f = aA$ | $\sigma_f^2 = a^2\sigma_A^2$ | $\sigma_f = \lvert a\rvert\sigma_A$ |
| $f = aA + bB$ | $\sigma_f^2 = a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}$ | $\sigma_f = \sqrt{a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}}$ |
| $f = aA - bB$ | $\sigma_f^2 = a^2\sigma_A^2 + b^2\sigma_B^2 - 2ab\,\sigma_{AB}$ | $\sigma_f = \sqrt{a^2\sigma_A^2 + b^2\sigma_B^2 - 2ab\,\sigma_{AB}}$ |
| $f = AB$ | $\sigma_f^2 \approx B^2\sigma_A^2 + A^2\sigma_B^2 + 2AB\,\sigma_{AB}$ | $\sigma_f \approx \sqrt{B^2\sigma_A^2 + A^2\sigma_B^2 + 2AB\,\sigma_{AB}}$ |
| $f = \frac{A}{B}$ | $\sigma_f^2 \approx f^2\left[\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 - 2\frac{\sigma_{AB}}{AB}\right]$ [11] | $\sigma_f \approx \lvert f\rvert\sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 - 2\frac{\sigma_{AB}}{AB}}$ |
| $f = aA^b$ | $\sigma_f^2 \approx \left(abA^{b-1}\sigma_A\right)^2 = \left(\frac{fb\sigma_A}{A}\right)^2$ | $\sigma_f \approx \left\lvert abA^{b-1}\sigma_A\right\rvert = \left\lvert\frac{fb\sigma_A}{A}\right\rvert$ |
| $f = a\ln(bA)$ | $\sigma_f^2 \approx \left(a\frac{\sigma_A}{A}\right)^2$ [12] | $\sigma_f \approx \left\lvert a\frac{\sigma_A}{A}\right\rvert$ |
| $f = a\log_{10}(A)$ | $\sigma_f^2 \approx \left(a\frac{\sigma_A}{A\ln(10)}\right)^2$ [12] | $\sigma_f \approx \left\lvert a\frac{\sigma_A}{A\ln(10)}\right\rvert$ |
| $f = ae^{bA}$ | $\sigma_f^2 \approx f^2\left(b\sigma_A\right)^2$ [13] | $\sigma_f \approx \lvert f\left(b\sigma_A\right)\rvert$ |
| $f = a^{bA}$ | $\sigma_f^2 \approx f^2\left(b\ln(a)\sigma_A\right)^2$ | $\sigma_f \approx \lvert f\left(b\ln(a)\sigma_A\right)\rvert$ |
| $f = A^B$ | $\sigma_f^2 \approx f^2\left[\left(\frac{B}{A}\sigma_A\right)^2 + \left(\ln(A)\sigma_B\right)^2 + 2\frac{B\ln(A)}{A}\sigma_{AB}\right]$ | $\sigma_f \approx \lvert f\rvert\sqrt{\left(\frac{B}{A}\sigma_A\right)^2 + \left(\ln(A)\sigma_B\right)^2 + 2\frac{B\ln(A)}{A}\sigma_{AB}}$ |

# Optimization of counting experiments

Measurement of a long-lived radioactive source in the present of a steady-state background

$S \equiv$ counting rate due to the source alone without background

$B \equiv$ counting rate due to background

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B}$$

$T_{S+B}$=measurement time for both background and source
$T_B$= just background

## Optimization of counting experiments

$$\sigma_s = \sqrt{\left(\frac{\sigma_{N_1}}{T_{S+B}}\right)^2 + \left(\frac{\sigma_{N_2}}{T_B}\right)^2}$$

$$\sigma_s = \sqrt{\left(\frac{N_1}{T_{S+B}^2}\right) + \left(\frac{N_2}{T_B^2}\right)}$$

$$\sigma_s = \sqrt{\left(\frac{S+B}{T_{S+B}}\right) + \left(\frac{B}{T_B}\right)}$$

How should we chose the fraction regarding $T_{S+B}$ and $T_B$ to get a minimum standard deviation?

$T = T_{S+B} + T_B$ (constant)

$$2\sigma_s d\sigma_s = -\frac{S+B}{T_{S+B}^2} dT_{S+B} - \frac{B}{T_B^2} dT_B$$

$d\sigma_s = 0$ *local min*

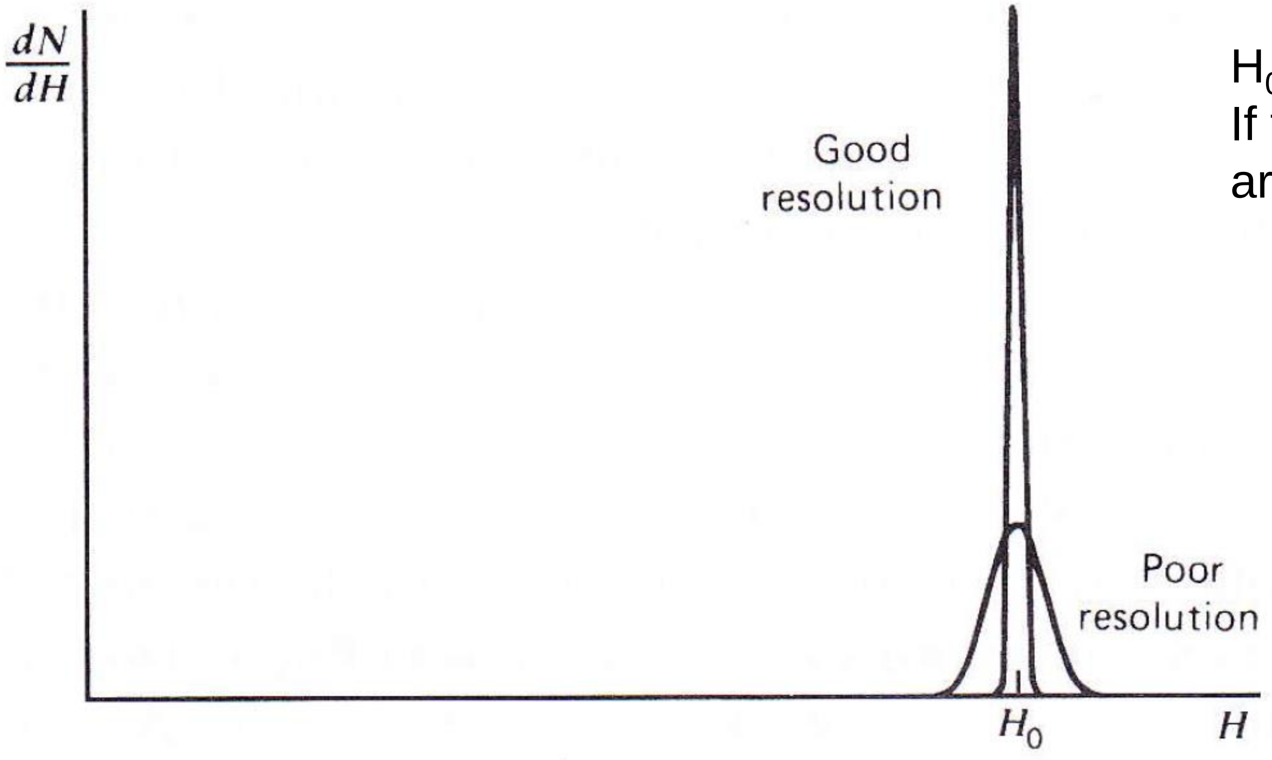
$dT_{S+B} = -dT_B$

$$\left.\frac{T_{S+B}}{T_B}\right|_{\text{opt}} = \sqrt{\frac{S+B}{B}}$$

How to chose T?

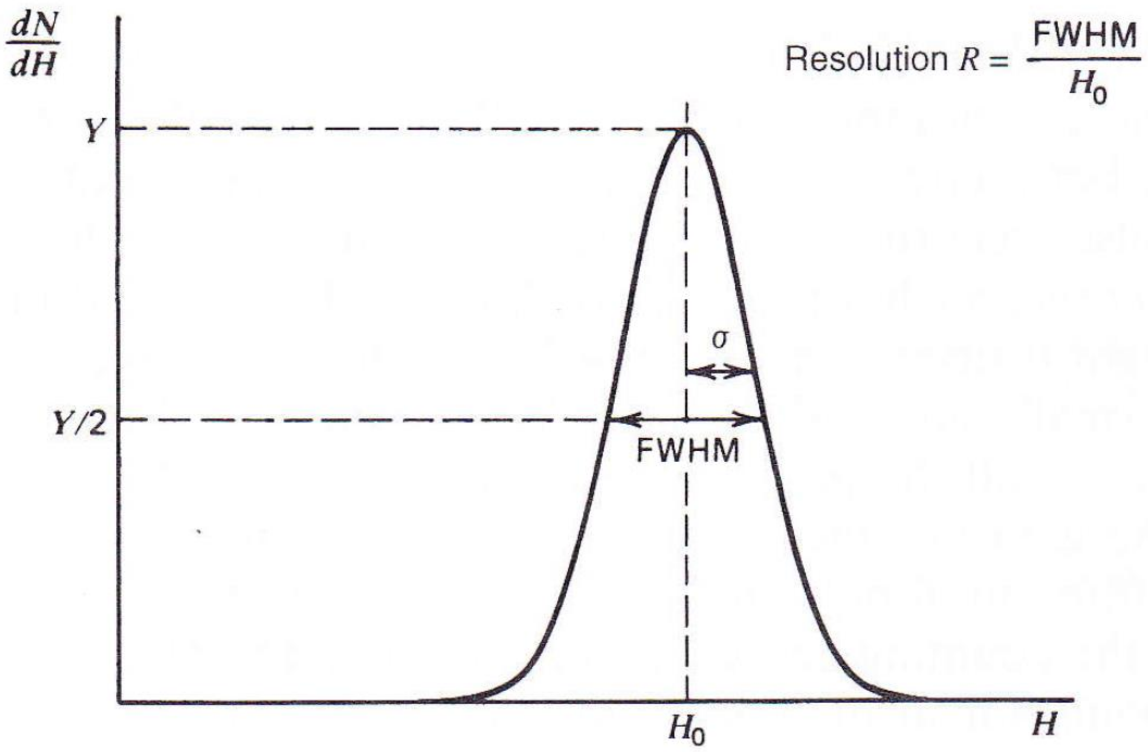
$$\epsilon \equiv \sigma_S / \breve{S} \qquad \frac{1}{T} = \epsilon^2 \frac{S^2}{\left(\sqrt{S+B} + \sqrt{B}\right)^2}$$

## Energy Resolution




$H_0$ is the average pulse height
If the same number of pulses are measured the are under the two graphs is equal

# Energy Resolution




For Normal distribution with the standard deviation of $\sigma$ the FWHM=2.35*$\sigma$

The energy resolution for a semiconductor detector could be less than 1%

Detector using scintillator show a energy resolution of 3-10%

## Energy Resolution

- If we assume the formation (in the detector) of each charge carrier is a Poisson process. Then a total number of N carrier generated in average, one would expect a standard deviation of $\sqrt{N}$.
- The average pulse amplitude $H_0 = KN$, $\sigma = K\sqrt{N}$ K is a proportionality constant
- $R_{\text{Poisson limit}} = \frac{FWHM}{H_0} = \frac{2.35K\sqrt{N}}{KN} = \frac{2.35}{\sqrt{N}}$ R better than 1%, N must be greater than 55 000
- Careful measurements on energy resolution on different radiation detectors have shown that the resolution is better than predicted with a factor of 3 to 4.
- The Fano Factor have been introduced to quantify the departure of this observation.

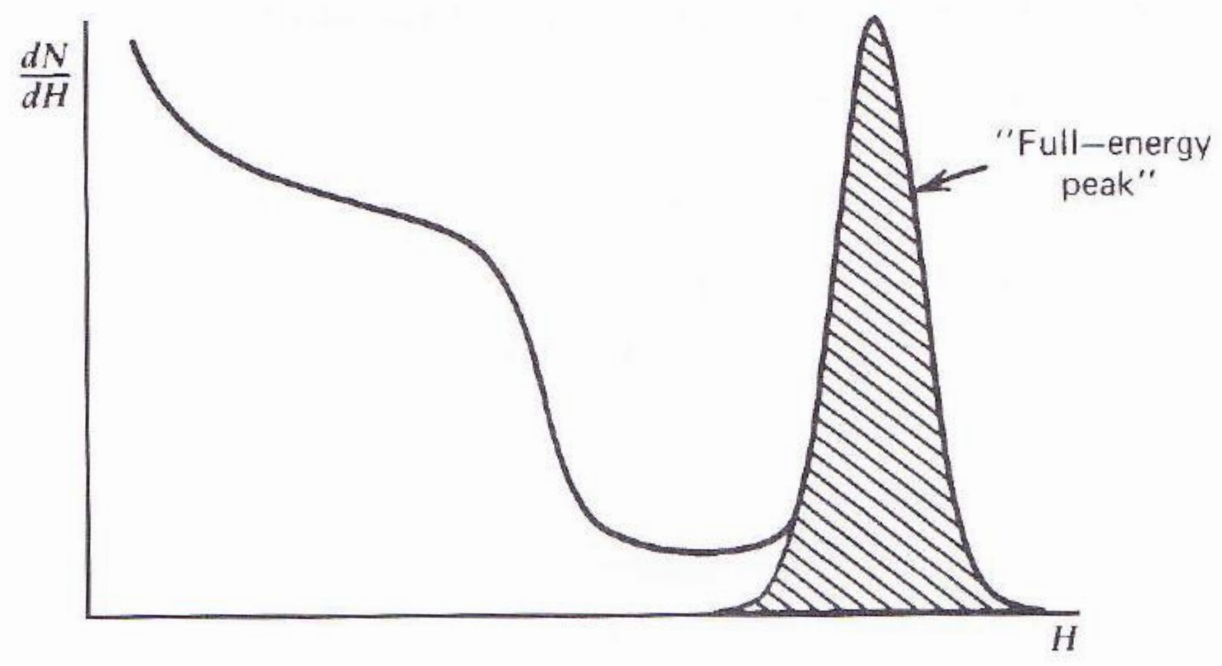
## Energy Resolution

- $R_{\text{statistical limit}} = \frac{2.35 K \sqrt{N} \sqrt{F}}{K N} = 2.35 \sqrt{\frac{F}{N}}$
- The sum of all losses in the detector must be equal to the initial particles energy, which is not an independent process, resulting that F must be less than 1
- $(\text{FWHM})^2_{\text{overall}} = (\text{FWHM})^2_{\text{statistical}} + (\text{FWHM})^2_{\text{electronic}} + \ldots\ldots\ldots$

## Detection Efficiency

- Absolute and intrinsic efficiency
- $\varepsilon_{abs} = \frac{number\ of\ pulses\ recorded}{number\ of\ radiation\ quanta\ emitted\ by\ source}$
- $\varepsilon_{int} = \frac{number\ of\ pulses\ recorded}{number\ of\ radiation\ quanta\ incident\ on\ detector}$
- For an isotropic source $\varepsilon_{abs} = \varepsilon_{int}(4\pi/\Omega)$ where $\Omega$ is the solid angle of the detector.

## Detection Efficiency



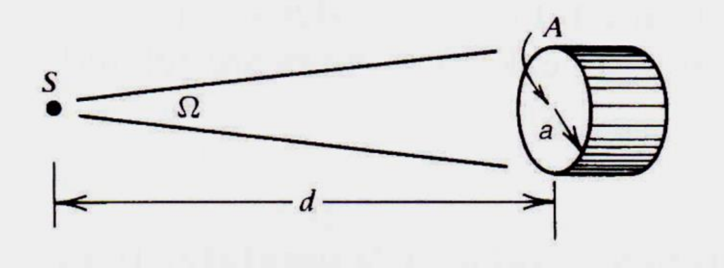

The intrinsic peak efficiency,

$$\varepsilon_{ip} = \frac{number\ of\ pulses\ recorded\ in\ the\ peak}{number\ of\ radiation\ quanta\ incident\ on\ detector}$$

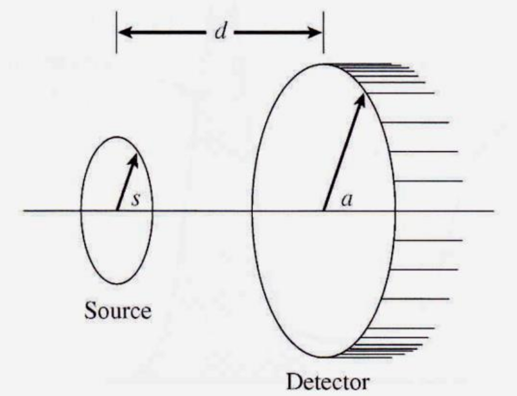
## Detection Efficiency

$$S = N \frac{4\pi}{\epsilon_{ip}\Omega}$$

- S is the number of emitted radioactive quanta from the source




$$\Omega = 2\pi \left(1 - \frac{d}{\sqrt{d^2 + a^2}}\right)$$



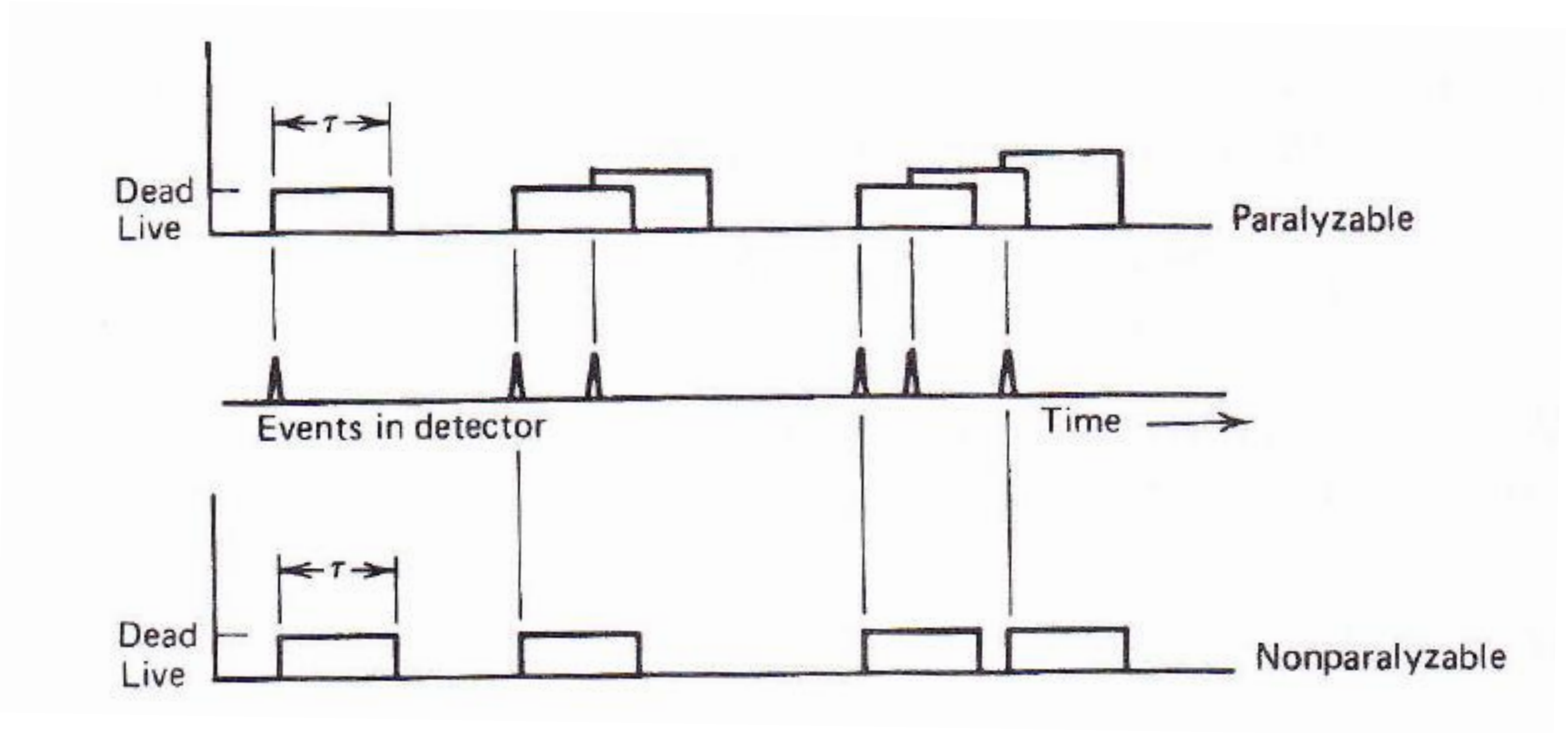

$$\Omega \cong 2\pi \left[1 - \frac{1}{(1+\beta)^{1/2}} - \frac{3}{8}\frac{\alpha\beta}{(1+\beta)^{5/2}} + \alpha^2[F1] - \alpha^3[F2]\right]$$

$$F1 = \frac{5}{16}\frac{\beta}{(1+\beta)^{7/2}} - \frac{35}{64}\frac{\beta^2}{(1+\beta)^{9/2}}$$

$$F2 = \frac{35}{128}\frac{\beta}{(1+\beta)^{9/2}} - \frac{315}{256}\frac{\beta^2}{(1+\beta)^{11/2}} + \frac{1155}{1024}\frac{\beta^3}{(1+\beta)^{13/2}}$$

$$\alpha = \left(\frac{s}{d}\right)^2 \quad \beta = \left(\frac{a}{d}\right)^2$$

# Dead Time




Two different models:

- Paralyzable
- Nonparalyzable

## Dead Time

- Nonparalyzable

$$n - m = nm\tau$$

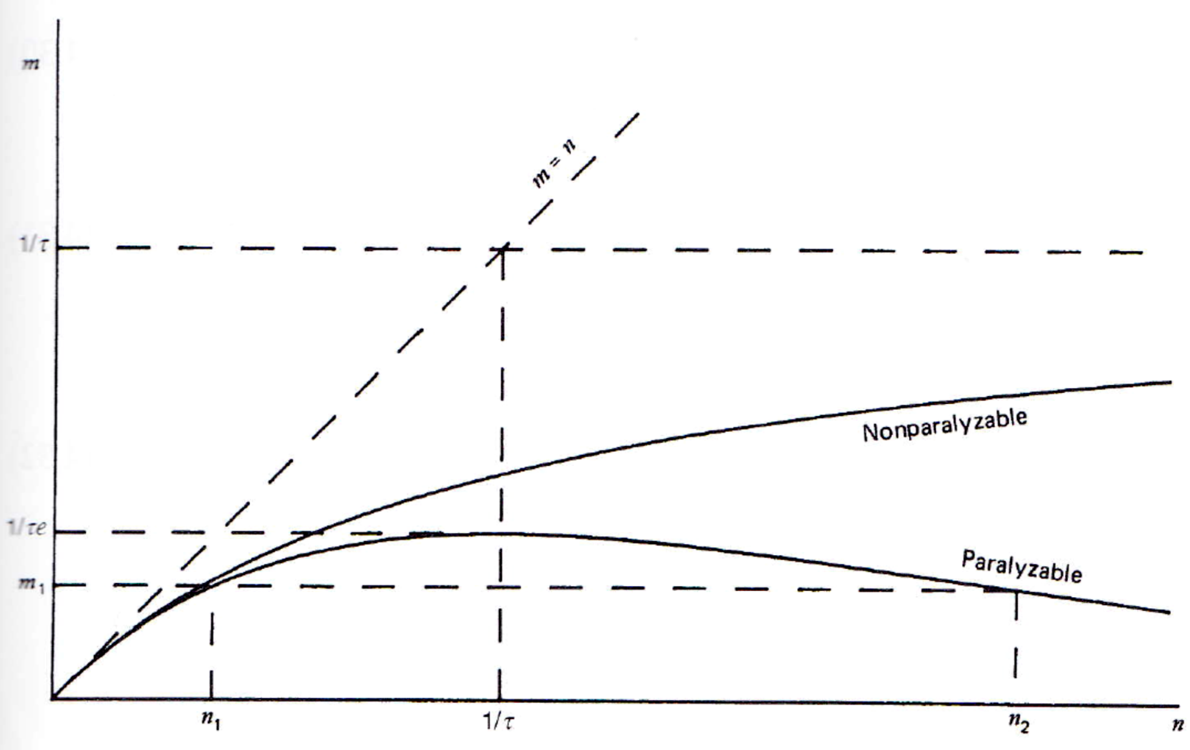
$n$ = true interaction rate
$m$ = recorded count rate
$\tau$ = system dead time

$$n = \frac{m}{1 - m\tau}$$

- Paralyzable

$$m = ne^{-n\tau}$$

Not possible to find n explicitly, must be solved numerical



# Dead Time

For low rates ($n \ll 1/\tau$) the following approximations can be written:

*Nonparalyzable* $$m = \frac{n}{1 + n\tau} \cong n(1 - n\tau)$$

*Paralyzable* $$m = ne^{-n\tau} \cong n(1 - n\tau)$$

Guideline: When losses are greater than 30-40% a system with smaller dead time should be used

# Dead Time

- Measurement of dead time, the two source method.
- Observing the count rate from two sources individual and in combination.
- The counting losses are nonlinear
  - Source 1 and source 2, assume a nonparalyzable system

$$n_{12} - n_b = (n_1 - n_b) + (n_2 - n_b)$$
$$n_{12} + n_b = n_1 + n_2$$

$n_1$, $n_2$, $n_{12}$, $n_b$ true counting rate, $n_b$ is the background

$$\frac{m_{12}}{1 - m_{12}\tau} + \frac{m_b}{1 - m_b\tau} = \frac{m_1}{1 - m_1\tau} + \frac{m_2}{1 - m_2\tau}$$

$m_1$, $m_2$, $m_{12}$, $m_b$ measured counting rate, $m_b$ is the background

$$X \equiv m_1 m_2 - m_b m_{12}$$
$$Y \equiv m_1 m_2 (m_{12} + m_b) - m_b m_{12} (m_1 + m_2)$$
$$Z \equiv \frac{Y(m_1 + m_2 - m_{12} - m_b)}{X^2}$$

$$\tau = \frac{X\left(1 - \sqrt{1 - Z}\right)}{Y}$$