RÄKNEÖVNING 6

5.5:28 Bestäm arean av området R definierat över $y=|x|$ och under $y=12-x^2$.

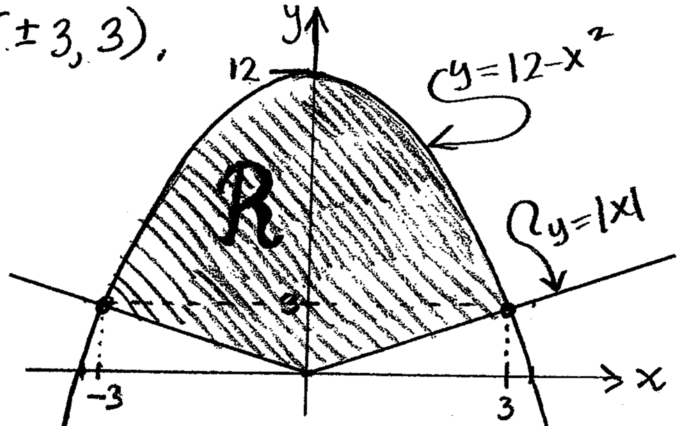
Lösning: Kurvorna $y=|x|$ och $y=12-x^2$ skär varandra där

$$|x| = 12-x^2 \Leftrightarrow \begin{cases} -x = 12-x^2, & x<0 \\ x = 12-x^2, & x\geq 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2-x-12=0, & x<0 \\ x^2+x-12=0, & x\geq 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \pm \sqrt{\frac{1}{4}+12} & \text{om } x<0 \\ x = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+12} & \text{om } x\geq 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \frac{1}{2} \pm \frac{7}{2} & \text{om } x<0 \\ x = -\frac{1}{2} \pm \frac{7}{2} & \text{om } x\geq 0 \end{cases} \Leftrightarrow \begin{cases} x = -3 & \text{om } x<0 \\ x = 3 & \text{om } x\geq 0 \end{cases}$$

d.v.s. skärningspunkter $(\pm 3, 3)$.



Vi ser att:

$$\text{area } R = \int_{-3}^{3}(12-x^2)\,dx - \int_{-3}^{3}|x|\,dx = [\text{jämna integrander}] =$$

$$= 2\int_{0}^{3}(12-x^2-x)\,dx = 2\left(12x-\tfrac{1}{3}x^3-\tfrac{1}{2}x^2\right)\Big|_0^3 =$$

$$= 2\left(\left(12\cdot 3-\tfrac{1}{3}\cdot 3^3-\tfrac{1}{2}\cdot 3^2\right)-\left(12\cdot 0-\tfrac{1}{3}0^3-\tfrac{1}{2}\cdot 0^2\right)\right) =$$

$$= 2\left(36-9-\tfrac{1}{2}9\right) = 72-18-9 = 45$$

5.5:46 Beräkna $H'(2)$ om $H(x) = 3x\int_{4}^{x^2} e^{-\sqrt{t}}\,dt$

Lösning: $H(x) = 3x \cdot \int_{4}^{x^2} e^{-\sqrt{t}}\,dt$

$$\Rightarrow H'(x) = 3\int_4^{x^2} e^{-\sqrt{t}}\,dt + 3x\frac{d}{dx}\int_4^{x^2} e^{-\sqrt{t}}\,dt =$$

$$= 3\int_4^{x^2} e^{-\sqrt{t}}\,dt + 3xe^{-\sqrt{x^2}}\frac{d}{dx}(x^2) =$$

$$= 3\int_4^{x^2} e^{-\sqrt{t}}\,dt + 3xe^{-|x|}\cdot 2x =$$

$$= 3\int_4^{x^2} e^{-\sqrt{t}}\,dt + 6x^2e^{-|x|}$$

$$\Rightarrow H'(2) = 3\int_4^{4} e^{-\sqrt{t}}\,dt + 6\cdot 4e^{-|2|} =$$

$$= 0 + 24e^{-2} = 24/e^2$$

5.6:30 Beräkna $\int \sec^6 x \tan^2 x\,dx = \int \frac{\tan^2 x}{\cos^6 x}\,dx$

Lösning: Eftersom $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$ så testar vi substitutionen $u = \tan x$. Vi har.

$$\int \frac{\tan^2 x}{\cos^6 x}\,dx = \int \frac{\tan^2 x}{\cos^4 x}\,\frac{dx}{\cos^2 x} \underset{\text{Trig.ettan}}{=} \int \frac{(\sin^2 x + \cos^2 x)^2}{\cos^4 x}\tan^2 x\,\frac{dx}{\cos^2 x} =$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x}{\cos^2 x}\right)^2 \tan^2 x\,\frac{dx}{\cos^2 x} =$$

$$= \int (\tan^2 x + 1)^2 \tan^2 x\,\frac{dx}{\cos^2 x} =$$

$$= \left[\begin{array}{l} u = \tan x \\ du = \frac{dx}{\cos^2 x}\end{array}\right] = \int (u^2+1)^2 u^2\,du =$$

$$= \int (u^4 + 2u^2 + 1)u^2\,du = \int (u^6 + 2u^4 + u^2)\,du =$$

$$= \frac{1}{7}u^7 + \frac{2}{5}u^5 + \frac{1}{3}u^3 + C =$$

$$= \frac{1}{7}\tan^7 x + \frac{2}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$$

6.1:20 Beräkna $\int_1^e \sin(\ln x)\,dx$.

Lösning: Låt $I = \int_1^e \sin(\ln x)\,dx$. Vi får

$$I = \int_1^e \sin(\ln x)\,dx = \int_1^e u\,dv \overset{P.I.}{=} uv\Big|_1^e - \int_1^e v\,du =$$

$$= \Big[\text{Låt } u = \sin(\ln x),\quad dv = dx$$
$$\text{d.v.s. } du = \cos(\ln x)\tfrac{1}{x}dx,\quad v = x\Big] =$$

$$= (\sin(\ln x)x)\Big|_1^e - \int_1^e x\cos(\ln x)\tfrac{1}{x}\,dx =$$

$$= \sin(\ln e)e - \underbrace{\sin(\ln 1)\cdot 1}_{=0} - \int_1^e \cos(\ln x)\,dx =$$

$$= e\sin 1 - \int_1^e \cos(\ln x)\,dx = e\sin 1 - \int_1^e u\,dv \overset{P.I.}{=}$$

$$= e\sin 1 - \left(uv\Big|_1^e - \int_1^e v\,du\right) =$$

$$= \Big[\text{Låt } u = \cos(\ln x),\quad dv = dx$$
$$du = -\sin(\ln x)\tfrac{1}{x}dx,\quad v = x\Big] =$$

$$= e\sin 1 - \left(\cos(\ln x)x\Big|_1^e - \int_1^e x(-\sin(\ln x))\tfrac{1}{x}\,dx\right) =$$

$$= e\sin 1 - \cos(\ln x)x\Big|_1^e - \int_1^e \sin(\ln x)\,dx =$$

$$= e\sin 1 - \left(\cos(\ln e)e - \underbrace{\cos(\ln 1)\cdot 1}_{=1}\right) - I =$$

$$= e\sin 1 - e\cos 1 + 1 - I, \text{ ger ekvation i } I.$$

Lös ut I: $\quad I = \frac{1}{2}\left[e(\cos 1 - \sin 1) + 1\right]$

6.2:28 Beräkna $\int \frac{d\theta}{\cos\theta(1+\sin\theta)}$.

Lösning: Vi kommer börja med substitutionen $u = \sin\theta$ för att få till en rationell integrand. Vi får:

$$\int \frac{d\theta}{\cos\theta(1+\sin\theta)} = \int \frac{\cos\theta \; d\theta}{\cos^2\theta(1+\sin\theta)} \overset{\substack{\text{Trig.-}\\ \text{ettan}}}{=} \int \frac{\cos\theta \, d\theta}{(1-\sin^2\theta)(1+\sin\theta)} =$$

$$= \begin{bmatrix} u = \sin\theta \\ du = \cos\theta d\theta \end{bmatrix} = \int \frac{du}{(1-u^2)(1+u)} =$$

$$= \int \frac{du}{((1+u)(1-u))(1+u)} = \int \frac{du}{(1-u)(1+u)^2} = (*)$$

Vi partialbråksuppdelar integranden:

$$\frac{1}{(1-u)(1+u)^2} \overset{\text{ansätt}}{=} \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{(1+u)^2} =$$

$$= \frac{A(1+u)^2 + B(1-u)(1+u) + C(1-u)}{(1-u)(1+u)^2} =$$

$$= \frac{A(1+2u+u^2) + B(1-u^2) + C - Cu}{(1-u)(1+u)^2} =$$

$$= \frac{(A-B)u^2 + (2A-C)u + (A+B+C)}{(1-u)(1+u)^2}$$

Vi får ekvationssystemet:

$$\begin{cases} A - B = 0 \\ 2A - C = 0 \\ A + B + C = 1 \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{matrix} (-2) & (-1) \\ & \\ & \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] (-1) \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \begin{matrix} \boxed{\tfrac{1}{2}} \\ \\ \boxed{\tfrac{1}{2}} \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right] (1/2) \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right] (1) \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \Leftrightarrow$$

$\Leftrightarrow A = B = 1/4, \; C = 1/2$

$$\Rightarrow (*) = \int \left(\frac{1/4}{1-u} + \frac{1/4}{1+u} + \frac{1/2}{(1+u)^2} \right) du =$$

$$= \frac{1}{4}\ln|1-u|\cdot(-1) + \frac{1}{4}\ln|1+u| - \frac{1}{2}\frac{1}{1+u} + C =$$
$$= \frac{1}{4}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{2(1+u)} + C =$$
$$= \frac{1}{4}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \frac{1}{2(1+\sin\theta)} + C$$

6.3:32 Beräkna $\int \frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}}\,dx$.

Lösning: Vi gör först en ordinär substitution $t=\sqrt{x^2+1}$. Sedan gör vi en omvänd substitution.

$$\int \frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}}\,dx = \begin{bmatrix} t=\sqrt{x^2+1} \Rightarrow x^2=t^2-1 \\ dt=\frac{x}{\sqrt{x^2+1}}\,dx \end{bmatrix} =$$
$$= \int\sqrt{2-(t^2-1)}\,dt = \int\sqrt{3-t^2}\,dt =$$
$$= \begin{bmatrix} t=\sqrt{3}\sin u \\ dt=\sqrt{3}\cos u\,du \end{bmatrix} = \int\sqrt{3-3\sin^2 u}\,\sqrt{3}\cos u\,du =$$
$$= 3\int\sqrt{1-\sin^2 u}\,\cos u\,du = 3\int\sqrt{\cos^2 u}\,\cos u\,du$$
$$= 3\int\cos^2 u\,du = 3\int\frac{1+\cos 2u}{2}\,du =$$
$$= \frac{3}{2}\left(u+\frac{1}{2}\sin 2u\right)+C = \frac{3}{2}(u+\sin u\cos u)+C =$$
$$= \frac{3}{2}\left(u+\sin u\sqrt{1-\sin^2 u}\right)+C = \left[\sin u = \frac{t}{\sqrt{3}}\right] =$$
$$= \frac{3}{2}\left(\arcsin\frac{t}{\sqrt{3}} + \frac{t}{\sqrt{3}}\sqrt{1-\left(\frac{t}{\sqrt{3}}\right)^2}\right)+C =$$
$$= \frac{3}{2}\left(\arcsin\frac{t}{\sqrt{3}} + \frac{t}{3}\sqrt{3-t^2}\right)+C = \left[t=\sqrt{x^2+1}\right] =$$
$$= \frac{3}{2}\left(\arcsin\sqrt{\frac{x^2+1}{3}} + \frac{1}{3}\sqrt{x^2+1}\sqrt{2-x^2}\right)+C =$$
$$= \frac{3}{2}\arcsin\sqrt{\frac{x^2+1}{3}} + \frac{1}{2}\sqrt{(x^2+1)(2-x^2)} + C$$

(Detta borde eg. testas p.g.a. alla kastade alt. tecken.) 5

6.5:20 Beräkna $\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx$.

Lösning: $\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx = \underbrace{\int_{-\infty}^{0} \frac{x}{1+x^4}\,dx}_{=I_1} + \underbrace{\int_{0}^{\infty} \frac{x}{1+x^4}\,dx}_{=I_2}$

Låt oss titta på I_2 först.

$$I_2 = \int_0^{\infty} \frac{x}{1+x^4}\,dx = \lim_{R\to\infty} \int_0^R \frac{x\,dx}{1+(x^2)^2} = \left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right] =$$

$$= \lim_{R\to\infty} \int_0^{R^2} \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\lim_{R\to\infty} \int_0^{R^2} \frac{du}{1+u^2} =$$

$$= \frac{1}{2}\lim_{R\to\infty} (\arctan u)\Big|_0^{R^2} = \frac{1}{2}\left(\lim_{R\to\infty} \arctan R^2 - 0\right) =$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

På samma sätt fås:

$$I_1 = \int_{-\infty}^{0} \frac{x}{1+x^4}\,dx = \lim_{R\to-\infty} \int_R^0 \frac{x\,dx}{1+(x^2)^2} = \ldots =$$

$$= \frac{1}{2}\lim_{R\to-\infty} (\arctan u)\Big|_{R^2}^{0} = \frac{1}{2}\left(0 - \lim_{R\to-\infty} \arctan R^2\right) =$$

$$= \frac{1}{2}\cdot\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx = I_1 + I_2 = \frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = 0$$