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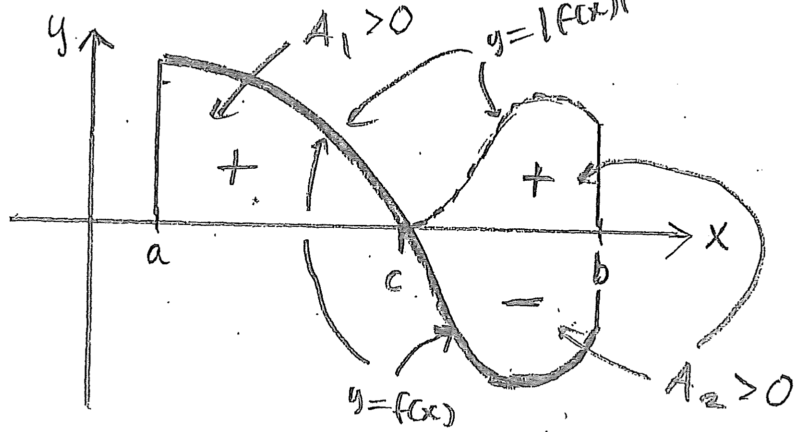
# Föreläsning 8

## Areaberäkningar

Integralen definieras ju som arean av ett område i xy-planet ($\mathbb{R}^2$ i flervariabelanalysspråk). Låt oss titta närmare på denna tillämpning.

Vi räknar arean av områden under x-axeln ($y<0$) som positiva trots att integralen är negativ där.

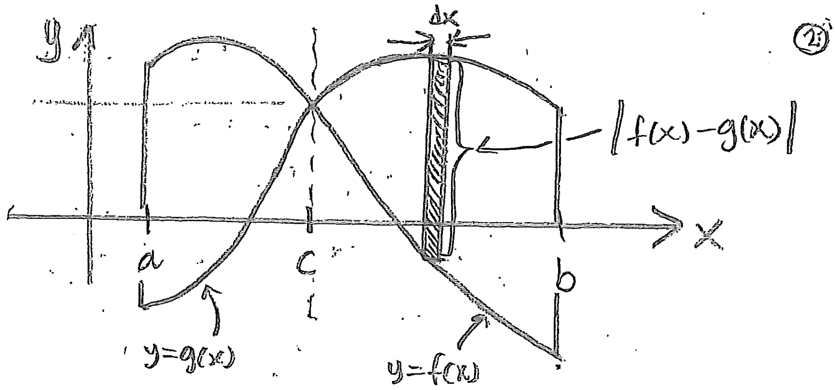



$\int_a^b |f(x)|\,dx$ ger alltid önskad area $A_1+A_2$

$$= \underbrace{\int_a^c f(x)\,dx}_{A_1} - \underbrace{\int_c^b f(x)\,dx}_{A_2}$$

Arean $A$ av område $R$ mellan $y=f(x)$ och $y=g(x)$ ges av formeln:

$A=\int_a^b |f(x)-g(x)|\,dx$ , se figur:




Detta eftersom arean hos infinitisemala rektangeln är $\underbrace{|f(x)-g(x)|}_{\text{höjd } (\geq 0)} \underbrace{dx}_{\text{bredd}}$ som summeras (eg. integreras) från a till b.

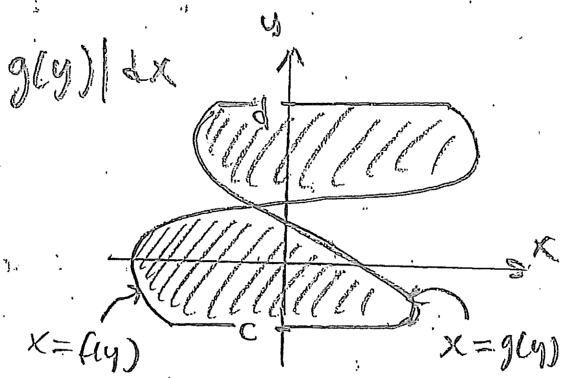
I figuren fås: $A = \int_a^c (f(x)-g(x))\,dx +$

$$+\int_c^b (g(x)-f(x))\,dx$$

Om $x=f(y)$, $x=g(y)$ så blir formeln

$$A = \int_c^d |f(y)-g(y)|\,dx$$

(Samma men $x \rightleftarrows y$ och $a \mapsto c$, $b \mapsto d$)




Exempel: Arean A av området R mellan $y = x^2 - 2x$ och $y = 4 - x^2$

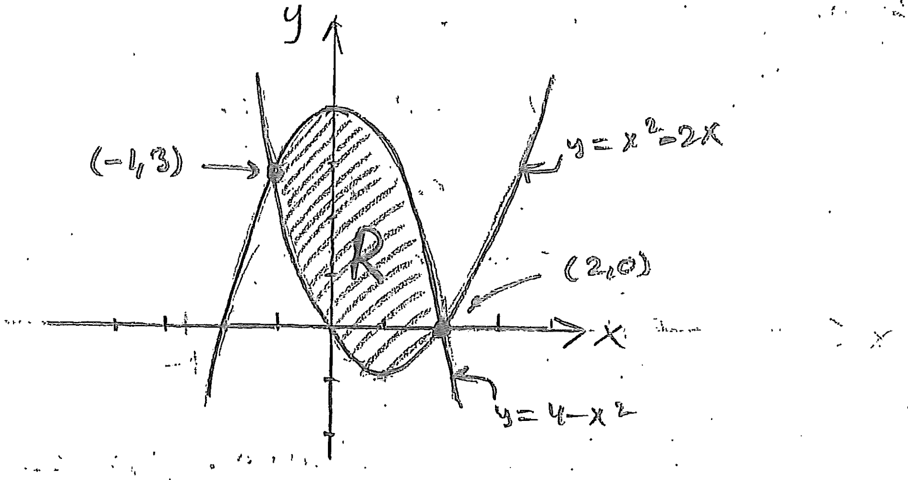
Lösning: Inga axlar nämnda så leta skärning mellan kurvorna: $x^2 - 2x = y = 4 - x^2 \Leftrightarrow$

③ $\Leftrightarrow 2x^2-2x-4=0 \Leftrightarrow x^2-x-2=0 \Leftrightarrow$

$\Leftrightarrow x = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2-(-2)} = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2} =$

$= \frac{1}{2} \pm \sqrt{\frac{9}{4}} = \frac{1}{2} \pm \frac{3}{2} = \begin{cases} 4/2 = 2 \\ -2/2 = -1 \end{cases}$

d.v.s. punkter $(-1,3)$ och $(2,0)$. Se figur:




$4-x^2 \geq x^2-2x \quad \forall x \in [-1,2]$

$\Rightarrow A = \int_{-1}^{2} \left((4-x^2)-(x^2-2x)\right) dx =$

$= \int_{-1}^{2} (4+2x-2x^2)\, dx =$ [Analysens huvudsats] $=$

$= \left(4x+x^2-\frac{2}{3}x^3\right)\Big|_{-1}^{2} =$

$= \left(4\cdot 2+2^2-\frac{2}{3}\cdot 2^3\right) - \left(4\cdot(-1)+(-1)^2-\frac{2}{3}(-1)^3\right) =$

$= 8+4-\frac{16}{3}+4-1-\frac{2}{3} = 15-\frac{18}{3} =$

$= 15-6 = \underline{\underline{9}}$ $\square$

Nu till frågan hur man tar hand om mer komplicerade integraler än de i gröna rutan på s. 317 (s. 302ff i uppl. 6)

# Integrationsmetoder

Memorera de s.k. elementära integralerna (d.v.s. antiderivatorna) på sid. 317 (s. 302ff i uppl. 6). Hur går vi vidare med dessa? Vi inför diverse integrationsmetoder:

- Substitution
- Partiell integrering
- Partialbråksuppdelning
- Invers substitution

**Substitution:** Metoden grundar sig på kedjeregeln.

**Sats:** Antag $g$ deriverbar på $[a,b]$ som uppfyller $g(a) = A$ och $g(b) = B$. Antag också att $f$ är kontinuerlig på $g$:s värdemängd. Då gäller:

$$\int_a^b f(g(x))g'(x)\,dx = \int_A^B f(u)\,du$$

**Bevis:** Låt $F$ vara antiderivata till $f$, d.v.s. $F'(u) = f(u)$.

$$\Rightarrow \frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

$\boxed{\text{kedjeregeln!}}$

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$$\Longrightarrow \int_a^b f(g(x))\,g'(x)\,dx = \int_a^b \frac{d}{dx} F(g(x))\,dx = \begin{bmatrix}\text{Analysens}\\ \text{huvud-}\\ \text{sats}\end{bmatrix}$$

$$= F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) =$$

$$= F(B) - F(A) = F(u)\Big|_A^B =$$

$$= [\text{Analysens huvudsats}] = \int_A^B \frac{d}{du} F(u)\,du =$$

$$= \int_A^B F'(u)\,du = \int_A^B f(u)\,du \qquad \square$$

<u>Notera:</u> Fungerar också med substitution i <u>obestämda</u> integraler. Då vill man behålla ursprungsvariabeln:

Obestämd: Inga angivna gränser

$$\underbrace{\int f(g(x))\,g'(x)\,dx}_{\text{antiderivatan till } f(g(x))g'(x)} = \left(\int f(u)\,du\right)\Big|_{u=g(x)}$$

<u>Exempel:</u>
(Obestämd integral, d.v.s. antiderivata)

$$\int e^x \sqrt{1+e^x}\,dx = \begin{bmatrix} u = 1+e^x & (=g(x)) \\ du = e^x dx & (=g'(x)\,dx) \end{bmatrix} =$$

$$= \int \sqrt{u}\,du = \int u^{1/2}\,du =$$

$$= \frac{2}{3} u^{3/2} + C = [\text{Sätt in } u = 1+e^x] =$$

$$= \boxed{\frac{2}{3}(1+e^x)^{3/2} + C} \qquad \square$$

<u>Exempel:</u>
(bestämd integral)

$$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx = \begin{bmatrix} u = \sqrt{x+1} & \cdots \\ du = \dfrac{1}{2\sqrt{x+1}}\,dx, \end{bmatrix} \begin{cases} x=8 \Leftrightarrow u=\sqrt{9}=3 \\ x=0 \Leftrightarrow u=\sqrt{1}=1 \end{cases}$$

$$= \int_1^3 \cos u \cdot 2\,du = 2\int_1^3 \cos u\,du =$$

$$= 2(\sin u)\Big|_1^3 = \boxed{2(\sin 3 - \sin 1)} \qquad \square$$

**Partiell integrering:** Metoden grundar sig på produktregeln.

Antag $U(x)$ och $V(x)$ deriverbara.

$$\Rightarrow \frac{d}{dx}\left(U(x)V(x)\right) \underset{\text{Produktregeln!}}{=} U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}$$

$$\Rightarrow \int \frac{d}{dx}\left(U(x)V(x)\right)dx = \int \left(U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}\right)dx$$

$$\Rightarrow U(x)V(x) = \int U(x)\frac{dV}{dx}dx + \int V(x)\frac{dU}{dx}dx$$

$$\Rightarrow \boxed{\int U(x)\frac{dV}{dx}dx = U(x)V(x) - \int V(x)\frac{dU}{dx}dx}$$

**Alternativt:** $\int \underset{\downarrow}{U}\, \underset{\uparrow}{dV} = UV - \int V\, dU$ (minnesregel)

**Exempel:** (obestämd)

$$\int \underbrace{x^2}_{\downarrow}\underbrace{\sin x\, dx}_{\uparrow} = \left[\begin{cases} U = x^2 \\ dU = 2x\,dx \end{cases}, \begin{cases} dV = \sin x\, dx \\ V = -\cos x \end{cases}\right] =$$

$$= x^2(-\cos x) - \int (-\cos x)\cdot 2x\,dx =$$

$$= -x^2\cos x + 2\int \underbrace{x}_{\downarrow}\underbrace{\cos x\, dx}_{\uparrow} =$$

$$= \left[\begin{cases} U = x \\ dU = dx \end{cases}, \begin{cases} dV = \cos x\, dx \\ V = \sin x \end{cases}\right] =$$

$$= -x^2\cos x + 2\left(x\sin x - \int \sin x\, dx\right) =$$

$$= \underline{\underline{-x^2\cos x + 2x\sin x + 2\cos x + C}} \quad \square$$

7 Exempel: (bestäm)

$$\int_1^e x^3(\ln x)^2\,dx = \int_1^e \underbrace{(\ln x)^2}_{\downarrow}\,\underbrace{x^3}_{\uparrow}dx =$$

$$= \left[\begin{cases} U = (\ln x)^2 \\ dU = 2\frac{\ln x}{x}\,dx \end{cases}, \begin{cases} dV = x^3dx \\ V = \frac{1}{4}x^4 \end{cases}\right] =$$

$$= \left((\ln x)^2\tfrac{1}{4}x^4\right)\Big|_1^e - \int_1^e \frac{1}{4}x^4\cdot 2\,\frac{\ln x}{x}\,dx =$$

$$= \left(\tfrac{1}{4}x^4(\ln x)^2\right)\Big|_1^e - \frac{1}{2}\int_1^e \underbrace{\ln x}_{\downarrow}\,\underbrace{x^3dx}_{\uparrow} =$$

$$= \left[\begin{cases} U = \ln x \\ dU = \frac{1}{x}dx \end{cases}, \begin{cases} dV = x^3dx \\ V = \frac{1}{4}x^4 \end{cases}\right] =$$

$$= \left(\tfrac{1}{4}x^4(\ln x)^2\right)\Big|_1^e - \frac{1}{2}\left(\left(\ln x\cdot\tfrac{1}{4}x^4\right)\Big|_1^e - \right.$$

$$\left. - \int_1^e \frac{1}{4}x^4\cdot\frac{1}{x}dx\right) =$$

$$= \left(\tfrac{1}{4}x^4(\ln x)^2\right)\Big|_1^e - \frac{1}{8}\left(x^4\ln x\right)\Big|_1^e +$$

$$+ \frac{1}{8}\int_1^e x^3\,dx =$$

$$= \left(\tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{8}x^4\ln x + \tfrac{1}{32}x^4\right)\Big|_1^e =$$

$$= \frac{1}{32}\left(x^4\left(8(\ln x)^2 - 4\ln x + 1\right)\right)\Big|_1^e =$$

$$= \frac{1}{32}\left(e^4(8(\ln e)^2 - 4\ln e + 1) - \right.$$

$$\left. - 1^4(8(\ln 1)^2 - 4\ln 1 + 1)\right) =$$

$$= \frac{1}{32}\left(e^4(8-4+1) - 1(0-0+1)\right) =$$

$$= \boxed{\frac{1}{32}\left(5e^4 - 1\right)}$$

□

**Partialbråksuppdelning:** Nu vill vi integrera funktioner av typen $\frac{P(x)}{Q(x)}$ där $P, Q$ är polynom. Kvoten kallas för rationell funktion.

Om $\frac{P(x)}{Q(x)}$ är en komplicerad kvot så kan man dela upp den i en summa av enklare kvoter genom s.k. partialbråksuppdelning. Denna åstadkoms m.h.a en ansättning.

**Exempel:** Beräkna $\int \frac{x+4}{x^2-5x+6}\,dx$

**Lösning:** Faktorisera nämnaren:

$$x^2-5x+6=0 \Leftrightarrow x=\frac{5}{2}\pm\sqrt{\left(\frac{5}{2}\right)^2-6}=$$

$$=\frac{5}{2}\pm\sqrt{\frac{25}{4}-\frac{24}{4}}=\frac{5}{2}\pm\frac{1}{2}=\begin{cases}3\\2\end{cases}$$

$$\Rightarrow x^2-5x+6=(x-2)(x-3)$$

$$\Rightarrow \frac{x+4}{x^2-5x+6}=\frac{x+4}{(x-2)(x-3)}=\text{[ansätt en partialbråksuppdelning]}=$$

$$=\frac{A}{x-2}+\frac{B}{x-3}=$$

$$=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}=$$

$$=\frac{(A+B)x+(-3A-2B)}{(x-2)(x-3)}$$

Identifiera täljarna:

$$x+4=(A+B)x+(-3A-2B)$$

$$\Rightarrow \begin{cases} A+B = 1 \\ -3A-2B = 4 \end{cases} \Leftrightarrow \begin{cases} B = 1-A \\ -3A-2(1-A) = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} B = 1-A \\ -A = 6 \end{cases} \Leftrightarrow \begin{cases} B = 1+6 = 7 \\ A = -6 \end{cases}$$

$$\Rightarrow \int \frac{x+4}{x^2-5x+6}\,dx = \int \left(\frac{-6}{x-2} + \frac{7}{x-3}\right) dx =$$

$$= -6\int \frac{dx}{x-2} + 7\int \frac{dx}{x-3} =$$

$$= \underline{\underline{-6 \ln|x-2| + 7\ln|x-3| + C}} \quad \square$$

Allmänt recept:

**Sats:** Låt $P$ och $Q$ vara reella polynom och antag att $P$:s grad är mindre än $Q$:s. Då gäller:

(a) Man kan alltid faktorisera $Q$ som

$$Q(x) = K(x-a_1)^{m_1}\cdots(x-a_j)^{m_j}\cdot (x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}$$

där $x^2+b_1x+c_1, \ldots, x^2+b_kx+c_k$ saknar reella rötter.

(b) Rationella funktionen $\frac{P(x)}{Q(x)}$ kan partialbråksuppdelas enligt:

(i) $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_m}{(x-a)^m}$

för faktorer i $Q$ av typen $(x-a)^m$

(ii) $\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \ldots + \frac{B_nx+C_n}{(x^2+bx+c)^n}$

$A_1, \ldots, A_m;\ B_1, \ldots, B_n$ och $C_1, \ldots, C_n$ fås genom ansättning för faktorer i $Q$ av typen $(x^2+bx+c)^n$

<u>Notera:</u> Den partialbråksuppdelade funktionen kan nu integreras med hjälp av metoder vi infört, t.ex. substitution.

## <u>Invers substitution:</u> Substitution "baklänges":

$$\int_a^b f(x)dx = \int_A^B f(g(u))g'(u)du$$

där $a=g(A)$, $b=g(B)$ $(A=g^{-1}(a), B=g^{-1}(b))$

Vi ersätter $x$ med något komplicerat, $g(u)$.

Finns inga allmänna regler, men man kan pröva:

- Integraler med $\sqrt{a^2-x^2}$ $(a>0)$:

  Låt $x=a\sin\theta \Leftrightarrow \theta=\arcsin\frac{x}{a}$

- Integraler med $\sqrt{x^2+a^2}$ eller $\frac{1}{x^2+a^2}$ $(a>0)$:

  Låt $x=a\tan\theta \Leftrightarrow \theta=\arctan\frac{x}{a}$

- Integraler med $\sqrt{x^2-a^2}$ $(a>0)$:

  Låt $x=\frac{a}{\cos\theta} \Leftrightarrow \theta=\arccos\frac{a}{x}$

  eller alternativt

  låt $x=\frac{a}{2}(e^u+e^{-u}) \Leftrightarrow u=\ln\left(\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right), x\geq 1$

  (Notera: $\frac{1}{2}(e^u+e^{-u})$ brukar skrivas $\cosh u$)
  "cosinus hyperbolicus"

- Integraler med $\sqrt{ax+b}$:

  Låt $x=\frac{1}{a}(u^2-b) \Leftrightarrow ax+b=u^2 \Leftrightarrow u=\pm\sqrt{ax+b}$

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- Integraler med kvoter av polynom i $\sin\theta$ & $\cos\theta$:

Låt $x = \tan\frac{\theta}{2} \Leftrightarrow \theta = 2\arctan x$

(T.ex. $\int \frac{1}{2+\cos\theta}\, d\theta$.)

Notera: $\frac{1}{2}(e^u + e^{-u}) \overset{\text{def}}{=} \cosh u$ ; $\frac{1}{2}(e^u - e^{-u}) \overset{\text{def}}{=} \sinh u$

cosinus hyperbolicus — sinus hyperbolicus

(Hyperboliska funktioner)

$\frac{d}{du}\cosh u = \sinh u$, $\frac{d}{du}\sinh u = \cosh u$

"Hyperboliska ettan": $\cosh^2 u - \sinh^2 u = 1$

**Exempel:** Beräkna $\int \frac{dx}{(1+9x^2)^2}$

**Lösning:**

$$\int \frac{dx}{(1+9x^2)^2} = \int \frac{dx}{(1+(3x)^2)^2} =$$

$$= \left[ \text{Av typen } \frac{1}{x^2+a^2} : \begin{cases} 3x = \tan\theta \\ 3dx = (1+\tan^2\theta)\,d\theta \end{cases} \right] =$$

$$= \int \frac{\frac{1}{3}(1+\tan^2\theta)\,d\theta}{(1+\tan^2\theta)^2} = \frac{1}{3}\int \frac{d\theta}{1+\tan^2\theta} =$$

$$= \frac{1}{3}\int \frac{d\theta}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{3}\int \frac{\cos^2\theta}{\cos^2\theta + \sin^2\theta}\, d\theta =$$

$$= \frac{1}{3}\int \cos^2\theta \, d\theta = \frac{1}{3}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta =$$

$$= \frac{1}{6}\int (1+\cos 2\theta)\, d\theta = \frac{1}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{6}(\theta + \sin\theta\cos\theta) + C =$$

$$= \frac{1}{6}\left(\theta + \frac{\sin\theta}{\cos\theta}\,\frac{\cos^2\theta}{\cos^2\theta + \sin^2\theta}\right) + C =$$

$$= \frac{1}{6}\left(\theta + \frac{\sin\theta}{\cos\theta}\,\frac{\cos^2\theta/\cos^2\theta}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}\right) + C =$$ (12)

$$= \frac{1}{6}\left(\theta + \tan\theta\,\frac{1}{1+\tan^2\theta}\right) + C = [3x = \tan\theta] =$$

$$= \frac{1}{6}\left(\arctan 3x + 3x\,\frac{1}{1+(3x)^2}\right) + C =$$

$$= \frac{1}{6}\left(\arctan 3x + \frac{3x}{1+9x^2}\right) + C =$$

$$= \boxed{\frac{1}{6}\arctan 3x + \frac{1}{2}\,\frac{x}{1+9x^2} + C} \qquad \square$$

Notera: Alternativt hade vi kunnat låta

$$x = a\sinh u = \frac{a}{2}\left(e^{u} - e^{-u}\right)$$

(Funkar ofta för $\sqrt{x^2+a^2}$ och $\frac{1}{x^2+a^2}$, $a > 0$.)

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# Några jämna uppgifter

**5.7:14** Arean $A$ hos området $R$ mellan $y=\frac{4x}{3+x^2}$ och $y=1$

Lösning: Skärning:

$$\frac{4x}{3+x^2}=y=1$$

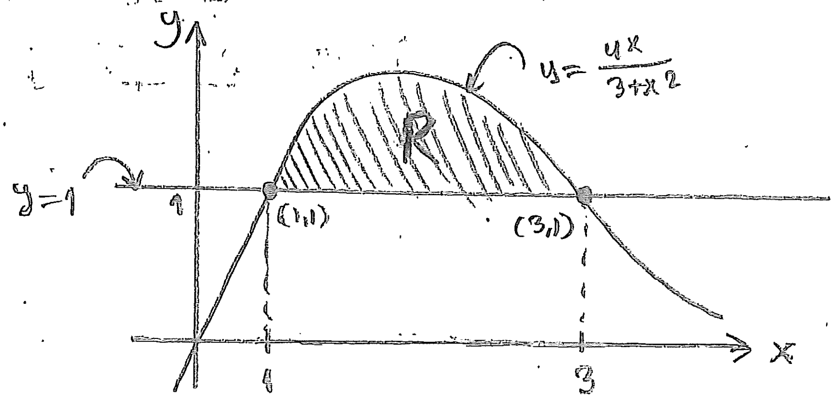
$$4x=3+x^2$$

$$x^2-4x+3=0$$

$$x=2\pm\sqrt{2^2-3}=2\pm1=\begin{cases}3\\1\end{cases}$$

$\Rightarrow$ Skärningspunkter $(1,1)$ och $(3,1)$

Låt oss göra en grov skiss:




Eftersom $\frac{4x}{3+x^2}\geq 1$ för $x\in[1,3]$ så fås:

$$A=\int_1^3\left(\frac{4x}{3+x^2}-1\right)dx=\int_1^3\frac{4x}{3+x^2}\,dx-\int_1^3 1\,dx=$$

$$=\left(2\ln|3+x^2|\right)\Big|_1^3-\left(x\right)\Big|_1^3=$$

$$=\left(2\ln|3+3^2|-2\ln|3+1^2|\right)-(3-1)=$$

$= 2\ln 12 - 2\ln 4 - 2 =$

$= 2\ln\frac{12}{4} - 2 = \underline{2\ln 3 - 2}$ (area-enheter)

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**5.6:22** $\displaystyle\int \frac{dx}{\sqrt{4+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x^2-2x)}} =$

$= \left[\begin{array}{l}\text{kvadrat-}\\\text{komplettera}\end{array}\right] = \displaystyle\int \frac{dx}{\sqrt{4-(x^2-2x+1)+1}} =$

$= \displaystyle\int \frac{dx}{\sqrt{5-(x-1)^2}} = \int \frac{dx}{\sqrt{5}\sqrt{1-\left(\frac{x-1}{\sqrt{5}}\right)^2}} =$

$= \left[\begin{cases} u = \frac{x-1}{\sqrt{5}} \\ du = \frac{1}{\sqrt{5}}dx \end{cases}\right] = \frac{1}{\sqrt{5}}\displaystyle\int \frac{\sqrt{5}\,du}{\sqrt{1-u^2}} =$

$= \displaystyle\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C =$

$= \boxed{\arcsin \frac{x-1}{\sqrt{5}} + C}$

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**6.1:26** $\displaystyle\int (\arcsin x)^2\,dx = \left[\begin{cases} \theta = \arcsin x \Leftrightarrow x = \sin\theta \\ dx = \cos\theta\, d\theta \end{cases}\right]$

$= \displaystyle\int \underset{\downarrow}{\theta^2}\, \underset{\uparrow}{\underline{\cos\theta\, d\theta}} =$

$\int U\,dV = UV - \int V\,dU$

$= \left[\begin{cases} U = \theta^2 \\ dU = 2\theta\, d\theta \end{cases} ; \begin{cases} dV = \cos\theta\, d\theta \\ V = \sin\theta \end{cases}\right] =$

$= \theta^2 \sin\theta - \displaystyle\int \sin\theta \cdot 2\theta\, d\theta =$

$= \theta^2 \sin\theta - 2\displaystyle\int \underset{\downarrow}{\theta}\, \underset{\uparrow}{\underline{\sin\theta\, d\theta}} =$

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$$= \left[ \begin{cases} u = \theta \\ du = d\theta \end{cases}, \begin{cases} dv = \sin\theta \, d\theta \\ v = -\cos\theta \end{cases} \right] =$$

$$= \theta^2 \sin\theta - 2\left(\theta(-\cos\theta) - \int(-\cos\theta)\,d\theta\right) =$$

$$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\int \cos\theta \, d\theta =$$

$$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C =$$

$$= \theta^2 \sin\theta + 2\theta\sqrt{1-\sin^2\theta} - 2\sin\theta + C =$$

$$= \underline{\underline{(\arcsin x)^2 x + 2(\arcsin x)\sqrt{1-x^2} - 2x + C}}$$

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## 6.2:16 $\int \frac{x^3+1}{12+7x+x^2}\,dx = (*)$

Polynomdivision av integranden:

$$\begin{array}{l} \quad x - 7 \\ \hline x^3 + 1 \quad \big| \ x^2+7x+12 \\ -x(x^2+7x+12) \qquad (= -x^3-7x^2-12x) \\ \hline \quad -7x^2-12x+1 \\ -(-7)(x^2+7x+12) \qquad (= 7x^2+49x+84) \\ \hline \qquad 37x+85 \qquad \leftarrow \text{restterm} \end{array}$$

$$\Rightarrow (*) = \int \left( x - 7 + \frac{37x+85}{x^2+7x+12} \right) dx = (**)$$

Faktorisera nämnaren i resttermen:

$$x^2+7x+12 = 0$$

$$x = -\frac{7}{2} \pm \sqrt{\frac{49}{4} - \frac{48}{4}} = -\frac{7}{2} \pm \frac{1}{2} = \begin{cases} -3 \\ -4 \end{cases}$$

$$\Rightarrow (**) = \int \left( x - 7 + \frac{37x+85}{(x+3)(x+4)} \right) dX =$$

$$= \frac{1}{2}x^2 - 7x + \int \frac{37x+85}{(x+3)(x+4)}\,dx = (\dagger)$$

Partialbråksuppdela:

$$\frac{37x+85}{(x+3)(x+4)} \overset{\text{ansätt}}{=} \frac{A}{x+3}+\frac{B}{x+4} = $$

$$= \frac{A(x+4)+B(x+3)}{(x+3)(x+4)} =$$

$$= \frac{(A+B)x+(4A+3B)}{(x+3)(x+4)}$$

$$\Longrightarrow \begin{cases} A+B=37 \\ 4A+3B=85 \end{cases} \Leftrightarrow \begin{cases} B=37-A \\ 4A+3(37-A)=85 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} B=37-A & =37-(-26)=63 \\ A=85-3\cdot 37=85-111=-26 \end{cases}$$

$$\Longrightarrow (\dagger) = \frac{1}{2}x^2-7x+\int\left(\frac{-26}{x+3}+\frac{63}{x+4}\right)dx =$$

$$= \frac{1}{2}x^2-7x-26\int\frac{dx}{x+3}+63\int\frac{dx}{x+4} =$$

$$= \boxed{\frac{1}{2}x^2-7x-26\ln|x+3|+63\ln|x+4|+C}$$

---

## 6.2:22 $\int \frac{x^2+1}{x^3+8}\,dx = (*)$

Vi ser att $x=-2$ rot till nämnaren (ty $(-2)^3+8=-2^3+8=-8+8=0$) så $x-(-2)=x+2$ faktor i $x^3+8$:

$$\begin{array}{l|l} x^3+8 & x+2 \\ \hline -x^2(x+2) & x^2-2x+4 \\ -2x^2+8 & \\ -(-2x)(x+2) & \\ 4x+8 & \\ -4(x+2) & \\ 0 & \end{array}$$

$(=-x^3-2x^2)$

$(=2x^2+4x)$

$(=-4x-8)$

$0 \leftarrow$ jämnt ut!

$$\Rightarrow x^3+8 = (x+2)(x^2-2x+4) =$$
$$= (x+2)\big((x^2-2x+1)+3\big) =$$
$$= (x+2)\big((x-1)^2+3\big)$$

$$\Rightarrow (*) = \int \frac{x^2+1}{(x+2)\big((x-1)^2+3\big)}\,dx = (**)$$

Partialbråksuppdela:

$$\frac{x^2+1}{(x+2)\big((x-1)^2+3\big)} \overset{\text{ansätt}}{=} \left[(x-1)^2+3 \text{ saknar nollställe}\right] =$$
$$= \frac{A}{x+2} + \frac{Bx+C}{(x-1)^2+3} =$$
$$= \frac{\big(A(x-1)^2+3A\big) + (x+2)(Bx+C)}{(x+2)\big((x-1)^2+3\big)} =$$
$$= \frac{(Ax^2-2Ax+4A)+\big(Bx^2+(2B+C)x+2C\big)}{(x+2)\big((x-1)^2+3\big)} =$$
$$= \frac{(A+B)x^2+(-2A+2B+C)x+(4A+2C)}{(x+2)\big((x-1)^2+3\big)}$$

$$\begin{cases} A + B = 1 \\ -2A+2B+C = 0 \\ 4A + 2C = 1 \end{cases} \Leftrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 2 & 1 & 0 \\ 4 & 0 & 2 & 1 \end{array}\right] \\ \begin{array}{ccc} A & B & C \end{array} \end{array} \; 2 \;\; -4 \; \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & 2 & -3 \end{array}\right] 1 \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 3 & -1 \end{array}\right] \boxed{1/3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & -1/3 \end{array}\right] -1 \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 7/3 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \boxed{1/4} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 7/12 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \begin{array}{c} \\ -1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/12 \\ 0 & 1 & 0 & 7/12 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} A = 5/12 \\ B = 7/12 \\ C = -1/3 \end{cases}$$

$$\Longrightarrow (**) = \int \left( \frac{5/12}{x+2} + \frac{(7/12)x - 1/3}{(x-1)^2+3} \right) dx =$$

$$= \frac{5}{12} \int \frac{dx}{x+2} + \frac{1}{12} \int \frac{7x-4}{(x-1)^2+3}\, dx =$$

$$= \frac{5}{12} \ln|x+2| + \frac{1}{12} \int \frac{7x-4}{(x-1)^2+3}\, dx =$$

$$= \left[ \begin{cases} u = x-1 \\ du = dx \end{cases} \right] =$$

$$= \frac{5}{12} \ln|x+2| + \frac{1}{12} \int \frac{7(u+1)-4}{u^2+3}\, du =$$

$$= \frac{5}{12} \ln|x+2| + \frac{1}{12} \int \frac{7u}{u^2+3}\, du + \frac{1}{4} \int \frac{du}{u^2+3} =$$

$$= \frac{5}{12} \ln|x+2| + \frac{7}{24} \int \frac{2u}{u^2+3}\, du + \frac{1}{12} \int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1} =$$

$$= \frac{5}{12} \ln|x+2| + \frac{7}{24} \ln|u^2+3| + \frac{1}{12} \int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1} =$$

$$= \left[ \begin{cases} v = \frac{u}{\sqrt{3}} \\ dv = \frac{du}{\sqrt{3}} \end{cases} \right] = \frac{5}{2} \ln|x+2| + \frac{7}{24} \ln\left((x-1)^2+3\right)$$

$$+ \frac{1}{12} \int \frac{\sqrt{3}\, dv}{v^2+1} =$$

$$= \frac{5}{2} \ln|x+2| + \frac{7}{24} \ln\left((x-1)^2+3\right) +$$

$$+ \frac{\sqrt{3}}{12} \arctan v + C =$$

$v = u/\sqrt{3} = (x-1)/\sqrt{3}$

(19)

$$= \boxed{\frac{5}{2}\ln|x+2| + \frac{7}{24}\ln(x^2-2x+4) + \frac{\sqrt{3}}{12}\arctan\frac{x-1}{\sqrt{3}} + C}$$

---

**6.3:30** $\displaystyle\int \frac{dx}{1+x^{1/3}} = \left[\begin{cases} x = u^3 \\ dx = 3u^2\,du \end{cases}\right] =$

$$= \int \frac{3u^2\,du}{1+(u^3)^{1/3}} = 3\int \frac{u^2}{1+u}\,du = (*)$$

Notera att: $u^2 = u^2+2u+1-2u-1 =$

$$= (u+1)^2 - (2u+1) =$$

$$= (u+1)^2 - 2(u+1) + 1$$

$$\Longrightarrow (*) = 3\int \frac{(u+1)^2 - 2(u+1) + 1}{u+1}\,du =$$

$$= 3\int \left((u+1) - 2 + \frac{1}{u+1}\right) du =$$

$$= 3\int \left(u - 1 + \frac{1}{u+1}\right) du =$$

$$= 3\left(\frac{1}{2}u^2 - u + \ln|u+1|\right) + C = [u = x^{1/3}] =$$

$$= \boxed{\frac{3}{2}x^{2/3} - 3x^{1/3} + 3\ln|x^{1/3}+1| + C}$$

---

Se även RÖ 6 HT09 där jag löst bl.a:

6.1:20, 6.2:28, 6.3:32

Lösningar till dessa återfinns också nedan.

**6.1:20** Beräkna $\int_1^e \sin(\ln x)\,dx$. 

**Lösning:** Låt $I = \int_1^e \sin(\ln x)\,dx$. Vi får

$$I = \int_1^e \sin(\ln x)\,dx = \int_1^e u\,dv \overset{P.I.}{=} uv\Big|_1^e - \int_1^e v\,du =$$

$$= \Big[\text{Låt } u = \sin(\ln x),\quad dv = dx$$
$$\text{d.v.s. } du = \cos(\ln x)\tfrac{1}{x}dx,\quad v = x\Big] =$$

$$= (\sin(\ln x)x)\Big|_1^e - \int_1^e x\cos(\ln x)\tfrac{1}{x}\,dx =$$

$$= \sin(\ln e)e - \underbrace{\sin(\ln 1)\cdot 1}_{=0} - \int_1^e \cos(\ln x)\,dx =$$

$$= e\sin 1 - \int_1^e \cos(\ln x)\,dx = e\sin 1 - \int_1^e u\,dv \overset{P.I.}{=}$$

$$= e\sin 1 - \left(uv\Big|_1^e - \int_1^e v\,du\right) =$$

$$= \Big[\text{Låt } u = \cos(\ln x),\quad dv = dx$$
$$du = -\sin(\ln x)\tfrac{1}{x}dx,\quad v = x\Big] =$$

$$= e\sin 1 - \left(\cos(\ln x)x\Big|_1^e - \int_1^e x(-\sin(\ln x))\tfrac{1}{x}\,dx\right) =$$

$$= e\sin 1 - \cos(\ln x)x\Big|_1^e - \int_1^e \sin(\ln x)\,dx =$$

$$= e\sin 1 - (\cos(\ln e)e - \underbrace{\cos(\ln 1)\cdot 1}_{=1}) - I =$$

$$= e\sin 1 - e\cos 1 + 1 - I, \text{ ger ekvation i } I.$$

Lös ut $I$: $I = \frac{1}{2}[e(\cos 1 - \sin 1) + 1]$

---

**6.2:28** Beräkna $\int \frac{d\theta}{\cos\theta(1+\sin\theta)}$.

**Lösning:** Vi kommer börja med substitutionen $u = \sin\theta$ för att få till en rationell integrand. Vi får:

(21) $\int \frac{d\theta}{\cos\theta(1+\sin\theta)} = \int \frac{\cos\theta \, d\theta}{\cos^2\theta(1+\sin\theta)} \overset{\text{Trig.-ettan}}{=} \int \frac{\cos\theta\, d\theta}{(1-\sin^2\theta)(1+\sin\theta)} =$

$= \left[\begin{array}{l} u = \sin\theta \\ du = \cos\theta d\theta \end{array}\right] = \int \frac{du}{(1-u^2)(1+u)} =$

$= \int \frac{du}{((1+u)(1-u))(1+u)} = \int \frac{du}{(1-u)(1+u)^2} = (*)$

Vi partialbråksuppdelar integranden:

$\frac{1}{(1-u)(1+u)^2} \overset{\text{ansätt}}{=} \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{(1+u)^2} =$

$= \frac{A(1+u)^2 + B(1-u)(1+u) + C(1-u)}{(1-u)(1+u)^2} =$

$= \frac{A(1+2u+u^2) + B(1-u^2) + C - Cu}{(1-u)(1+u)^2} =$

$= \frac{(A-B)u^2 + (2A-C)u + (A+B+C)}{(1-u)(1+u)^2}$

Vi får ekvationssystemet:

$\begin{cases} A - B = 0 \\ 2A - C = 0 \\ A + B + C = 1 \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} (-2)\ (-1) \\ \\ \\ \end{array} \sim$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] (-1) \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} \\ \boxed{\tfrac{1}{2}} \\ \boxed{\tfrac{1}{2}} \end{array} \sim$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right] (1/2) \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right] (1) \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \Leftrightarrow$

$\Leftrightarrow A = B = 1/4, \quad C = 1/2$

$\Longrightarrow (*) = \int \left( \frac{1/4}{1-u} + \frac{1/4}{1+u} + \frac{1/2}{(1+u)^2} \right) du =$

$$= \frac{1}{4}\ln|1-u|\cdot(-1) + \frac{1}{4}\ln|1+u| - \frac{1}{2}\,\frac{1}{1+u} + C =$$

$$= \frac{1}{4}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{2(1+u)} + C =$$

$$= \frac{1}{4}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \frac{1}{2(1+\sin\theta)} + C$$

---

**6.3:32** Beräkna $\int \frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}}\,dx$.

Lösning: Vi gör först en ordinär substitution $t=\sqrt{x^2+1}$. Sedan gör vi en omvänd substitution.

$$\int \frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}}\,dx = \begin{bmatrix} t=\sqrt{x^2+1} \Rightarrow x^2=t^2-1 \\ dt=\frac{x}{\sqrt{x^2+1}}\,dx \end{bmatrix} =$$

$$= \int \sqrt{2-(t^2-1)}\,dt = \int\sqrt{3-t^2}\,dt =$$

$$= \begin{bmatrix} t=\sqrt{3}\sin u \\ dt=\sqrt{3}\cos u\,du \end{bmatrix} = \int\sqrt{3-3\sin^2 u}\,\sqrt{3}\cos u\,du =$$

$$= 3\int\sqrt{1-\sin^2 u}\,\cos u\,du = 3\int\sqrt{\cos^2 u}\,\cos u\,du$$

$$= 3\int\cos^2 u\,du = 3\int\frac{1+\cos 2u}{2}\,du =$$

$$= \frac{3}{2}\left(u+\frac{1}{2}\sin 2u\right)+C = \frac{3}{2}(u+\sin u\cos u)+C =$$

$$= \frac{3}{2}\left(u+\sin u\sqrt{1-\sin^2 u}\right)+C = \left[\sin u = \frac{t}{\sqrt{3}}\right] =$$

$$= \frac{3}{2}\left(\arcsin\frac{t}{\sqrt{3}} + \frac{t}{\sqrt{3}}\sqrt{1-\left(\frac{t}{\sqrt{3}}\right)^2}\right)+C =$$

$$= \frac{3}{2}\left(\arcsin\frac{t}{\sqrt{3}} + \frac{t}{3}\sqrt{3-t^2}\right)+C = \left[t=\sqrt{x^2+1}\right] =$$

$$= \frac{3}{2}\left(\arcsin\sqrt{\frac{x^2+1}{3}} + \frac{1}{3}\sqrt{x^2+1}\sqrt{2-x^2}\right)+C =$$

$$= \frac{3}{2}\arcsin\sqrt{\frac{x^2+1}{3}} + \frac{1}{2}\sqrt{(x^2+1)(2-x^2)} + C$$

(Detta borde eg. testas p.g.a. alla, kastade alt. tecken.)