Mittuniversitetet
NAT, Stefan Borell

Svenska

Tentamen i matematik
Fördjupningskurs i analys (MA092G/MA093G)
25 augusti 2011
Skrivtid: 5 timmar
Hjälpmedel: Godkänd miniräknare samt bifogad formelsamling

Till alla uppgifter skall fullständiga lösningar lämnas. Resonemang, ekvationslösningar och uträkningar får inte vara så knapphändiga att de blir svåra att följa. En uppgift per blad, skriv endast på en sida.

Betyg sätts efter hur väl lärandemålen är uppfyllda. Riktvärde för betygen är: A 22p, B 18p, C 14p, D 10p, E 9p.

1. (a) Ge den formella definitionen av påståendet "gränsvärdet av $f(x)$ är lika med L när x går mot a", det vill säga, definiera vad uttrycket $\lim_{x\to a} f(x) = L$ betyder. (1p)

   (b) Genom att endast använda den formella definitionen av gränsvärde, visa att
$$\lim_{x\to 1} \frac{1}{x-2} = -1. \qquad (2p)$$

2. (a) Ge den formella definitionen av påståendet "funktionen $f(x)$ är kontinuerlig i punkten $x = a$". (1p)

   (b) För vilka värden på k är funktionen
$$f(x) = \begin{cases} x+1 & x < 1 \\ (x-k)^2 & x \geq 1 \end{cases} \qquad (x \in \mathbb{R})$$
   kontinuerlig i punkten $x = 1$? (2p)

3. Låt
$$f(x) = \frac{x^3 - 3x^2 + 2x}{x^2 - x - 2}$$
   och avgör om följande gränsvärden existerar. Hitta gränsvärdet i de fall ett gränsvärde existerar.

   (a) $\lim_{x\to 0} f(x)$ (1p)

   (b) $\lim_{x\to 2} f(x)$ (1p)

   (c) $\lim_{x\to \infty} f(x)$ (1p)

4. En rektangulär reklamtavla skall utgöras av 100 m^2 yta med tryck, 2 m marginal i över- och underkant samt 4 m marginal längs med sidorna. Hitta de yttre sidlängderna på reklamtavlan som gör den totala arean minimal. (3p)

5. Låt $a < b$. Visa att derivatan av $f(x) = (x-a)^m(x-b)^n$ försvinner i någon punkt c i intervallet (a, b) om m och n är positiva heltal. (3p)

6. Betrakta funktionen $f(x) = \sin x$.

   (a) Bestäm det generella uttrycket av Taylors formel för f i punkten $a = 0$ med Lagranges restterm. (1p)

   (b) Bestäm en tillräckligt stor ordning n sådan att motsvarande Taylorpolynom ger en approximering av sinus för 1 radian som har 5 korrekta decimaler. Approximera $\sin 1$ genom att använda Taylorpolynomet av denna ordning. (2p)

7. (a) Lös initialvärdesproblemet

$$2y' = xy, \qquad y(0) = 1. \qquad (2p)$$

   (b) Använd den förbättrade Euler-metoden med $h = 1$ för att approximera värdet $y(2)$ av lösningen till ekvationen ovan. Kom ihåg att iterationerna som används i den förbättrade Euler-metoden för att approximera lösningen till ekvationen $y' = F(x, y)$, $y(x_0) = y_0$, är på formen

$$\begin{cases} x_{n+1} = x_n + h, \\ u_{n+1} = y_n + h \cdot F(x_n, y_n), \\ y_{n+1} = y_n + h \cdot \big(F(x_n, y_n) + F(x_{n+1}, u_{n+1})\big)/2. \end{cases} \qquad (1p)$$

8. (a) Approximera integralen

$$\int_1^3 xe^{-x^2}\,dx$$

   genom att använda Trapetsregeln för $h = 0{,}25$. Kom ihåg att Trapetsregeln för n steg ges av

$$T_n = h \cdot \left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right),$$

   där $x_j = x_0 + j \cdot h$, $0 \le j \le n$. (1p)

   (b) Är integralen

$$\int_1^\infty xe^{-x^2}\,dx$$

   konvergent eller divergent? Bestäm dess värde om den är konvergent eller förklara varför den divergerar. (2p)

Mid Sweden University
NAT, Stefan Borell

Engelska

Mathematics Examination
Calculus II (MA092G/MA093G)
25 August 2011
Duration: 5 hours
Aids permitted: An approved calculator and the formula collection attached

Working must be shown in full in order to obtain full credit for an exercise. Do one question per page and write on one side of the paper only.

The grade is determined by the extent to which a candidate demonstrates that the learning outcomes of the course have been met. Guide values for grades are: A 22p, B 18p, C 14p, D 10p, E 9p.

1. (a) Give the formal definition of the statement "the limit of $f(x)$ is equal to L as x approaches a," that is, define what the expression $\lim_{x \to a} f(x) = L$ means. (1p)

   (b) Using only the formal definition of limits, show that
   $$\lim_{x \to 1} \frac{1}{x-2} = -1. \qquad (2p)$$

2. (a) Give the formal definition of the statement "the function $f(x)$ is continuous at the point $x = a$." (1p)

   (b) For which values of k is the function
   $$f(x) = \begin{cases} x+1 & x < 1 \\ (x-k)^2 & x \geq 1 \end{cases} \qquad (x \in \mathbb{R})$$
   continuous at the point $x = 1$? (2p)

3. Let
   $$f(x) = \frac{x^3 - 3x^2 + 2x}{x^2 - x - 2}$$
   and determine whether the following limits exist. If a limit exists, find its value.

   (a) $\lim_{x \to 0} f(x)$ (1p)

   (b) $\lim_{x \to 2} f(x)$ (1p)

   (c) $\lim_{x \to \infty} f(x)$ (1p)

4. A rectangular billboard is to be made with 100 m^2 of printed area and with margins of 2 m at the top and bottom and 4 m on each side. Find the outside dimensions of the billboard if its total area is to be a minimum.

   (3p)

5. Let $a < b$. Show that the derivative of $f(x) = (x-a)^m(x-b)^n$ vanishes at some point c in the interval (a, b) if m and n are positive integers. (3p)

6. Consider the function $f(x) = \sin x$.

   (a) Find the general form of Taylor's formula for f at the point $a = 0$ with Lagrange remainder. (1p)

   (b) Find a sufficiently large order n for which the corresponding Taylor polynomial approximation will give the sine of 1 radian correct to 5 decimal places. Approximate $\sin 1$ by using the Taylor polynomial of this order. (2p)

7. (a) Solve the initial value problem

$$2y' = xy, \qquad y(0) = 1. \tag{2p}$$

   (b) Use Euler's Improved Method with $h = 1$ in order to approximate the value $y(2)$ of the solution to the equation above. Recall that the iterations used in Euler's Improved Method for approximating the solution the equation $y' = F(x, y)$, $y(x_0) = y_0$, are of the form

$$\begin{cases} x_{n+1} = x_n + h, \\ u_{n+1} = y_n + h \cdot F(x_n, y_n), \\ y_{n+1} = y_n + h \cdot \big(F(x_n, y_n) + F(x_{n+1}, u_{n+1})\big)/2. \end{cases} \tag{1p}$$

8. (a) Approximate the integral

$$\int_1^3 xe^{-x^2}\, dx$$

   by use of the Trapezoid Rule with $h = 0.25$. Recall that the Trapezoid rule for n steps is given by

$$T_n = h \cdot \left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right),$$

   where $x_j = x_0 + j \cdot h$, $0 \le j \le n$. (1p)

   (b) Is the integral

$$\int_1^\infty xe^{-x^2}\, dx$$

   convergent or divergent? Find its value if it is convergent or explain why it diverges. (2p)

Mid Sweden University
NAT, Stefan Borell

"Solutions" of the Exam Problems, Aug 25, 2011
(Some details may have been left out)

1. (a) Give the formal definition of the statement "the limit of $f(x)$ is equal to L as x approaches a," that is, define what the expression $\lim_{x \to a} f(x) = L$ means.

   (b) Using only the formal definition of limits, show that

$$\lim_{x \to 1} \frac{1}{x-2} = -1.$$

Solution. a) For any real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$0 < |x - a| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon.$$

b) We need to demonstrate that for any real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$0 < |x - 1| < \delta \qquad \Longrightarrow \qquad \left| \frac{1}{x-2} - (-1) \right| < \varepsilon.$$

Since we have division by $x - 2$ in the above, we have to assure that $x \neq 2$. Suppose that we require $\delta \leq 1/2$. Then $|x - 1| < \delta \leq 1/2$ implies $1/2 < x < 3/2$, which guarantees $x \neq 2$. Let us now examine the absolute value on the right-hand side of the implication above. Assuming that $0 < |x - 1| < \delta \leq 1/2$, we get

$$\left| \frac{1}{x-2} - (-1) \right| = \left| \frac{1}{x-2} + 1 \right| = \left| \frac{1}{x-2} + \frac{x-2}{x-2} \right|$$
$$= \left| \frac{x-1}{x-2} \right| = |x-1| \cdot \frac{1}{|x-2|} < \delta \cdot \frac{1}{|x-2|}.$$

Observe that $1/2 < x < 3/2$ implies $-3/2 < x - 2 < -1/2$, and therefore $1/2 < |x - 2| < 3/2$. This gives

$$\left| \frac{1}{x-2} - (-1) \right| < \delta \cdot \frac{1}{|x-2|} < \delta \cdot 2.$$

If we choose δ equal to the smallest of the two numbers $\varepsilon/2$ and $1/2$, so that $\delta \leq \varepsilon/2$ and $\delta \leq 1/2$, then we get

$$\left| \frac{1}{x-2} - (-1) \right| < 2\delta \leq \varepsilon.$$

This proves that if δ is equal to the smallest of the two numbers $\varepsilon/2$ and $1/2$, then the implication

$$0 < |x - 1| < \delta \qquad \Longrightarrow \qquad \left| \frac{1}{x-2} - (-1) \right| < \varepsilon$$

holds true.

2. (a) Give the formal definition of the statement "the function $f(x)$ is continuous at the point $x = a$."

(b) For which values of k is the function

$$f(x) = \begin{cases} x+1 & x < 1 \\ (x-k)^2 & x \geq 1 \end{cases} \qquad (x \in \mathbb{R})$$

continuous at the point $x = 1$?

Solution. a) $\lim_{x \to a} f(x) = f(a)$

b) We wish to find the values of k for which

$$\lim_{x \to 1} f(x) = f(1) = (1-k)^2.$$

Recall that the limit $\lim_{x \to 1} f(x)$ exists if and only if $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x)$. Let us determine these one-sided limits. For $x < 1$ we have $f(x) = x + 1$ and therefore

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x + 1 = 1 + 1 = 2.$$

For $x > 1$ we have $f(x) = (x-k)^2$ and therefore

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (x-k)^2 = (1-k)^2 = f(1).$$

Since $\lim_{x \to 1+} f(x) = f(1)$, it is enough to show that

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x).$$

By the above, this means that we are looking for the values of k such that $2 = (1-k)^2$. We get

$$(1-k)^2 = 2 \qquad \Longleftrightarrow \qquad 1 - k = \pm\sqrt{2}$$

which gives the two solutions $k = 1 \pm \sqrt{2}$. We conclude that the function $f(x)$ is continuous at the point $x = 1$ if and only if $k = 1 \pm \sqrt{2}$.

3. Let

$$f(x) = \frac{x^3 - 3x^2 + 2x}{x^2 - x - 2}$$

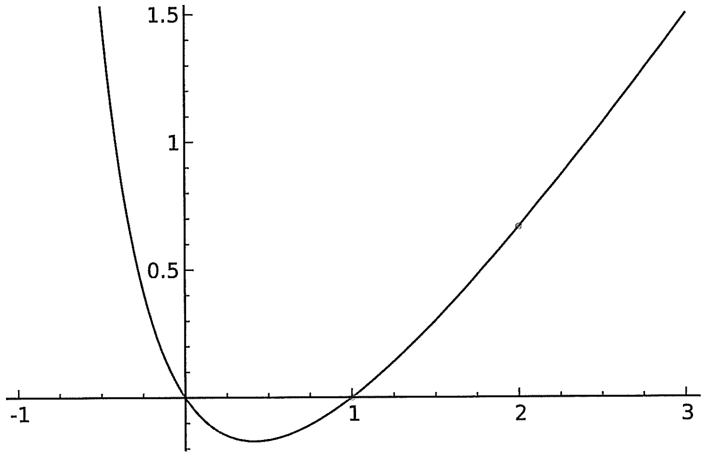
and determine whether the following limits exist. If a limit exists, find its value.

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to \infty} f(x)$

Solution. First we observe that f is a rational function. Thus, f is continuous at each point of the real line where its denominator is non-zero.



a) Observe that the denominator is non-zero for $x = 0$. Thus $f(x)$ is continuous at $x = 0$ and we get

$$\lim_{x \to 0} f(x) = f(0) = \frac{0^3 - 3 \cdot 0^2 + 2 \cdot 0}{0^2 - 0 - 2} = 0.$$

b) Observe that the denominator is zero for $x = 2$. In fact, we may write the denominator as

$$x^2 - x - 2 = (x + 1)(x - 2).$$

Moreover, the nominator is zero for $x = 2$ and

$$x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x - 1)(x - 2).$$

This means that, for $x \neq 2$, we have

$$f(x) = \frac{x^3 - 3x^2 + 2x}{x^2 - x - 2} = \frac{x(x-1)(x-2)}{(x+1)(x-2)} = \frac{x(x-1)}{x+1}$$

Since the right-hand side above is continuous at $x = 2$ we get

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x(x-1)}{x+1} = \frac{2(2-1)}{2+1} = \frac{2}{3}.$$

c) For $x \neq 0$ we have

$$f(x) = \frac{x^3 - 3x^2 + 2x}{x^2 - x - 2} = \frac{x^3(1 - 3/x^2 + 2/x^3)}{x^2(1 - 1/x - 2/x^2)} = x \cdot \frac{1 - 3/x^2 - 2/x^3}{1 - 1/x - 2/x^2}.$$

Consider the fraction at the right-hand side. Observe that nominator and denominator both tend to 1 as $x \to \infty$. This means that the fraction on the right-hand side tends to 1 as $x \to \infty$. Thus, as $x \to \infty$ it seems that $f(x)$ tends to ∞. Let us prove this in detail.

Recall that

$$\lim_{x\to\infty} f(x) = \infty$$

if and only if for any real number M there exists another real number N such that

$$x > N \quad \Longrightarrow \quad f(x) > M.$$

Since the fraction on the right-hand side above tends to 1 there is a real number R such that

$$\frac{1 - 3/x^2 - 2/x^3}{1 - 1/x - 2/x^2} > \frac{1}{2}$$

whenever $x > R$. Let N be a positive number such that $N > 2M$ and $N > R$. Let us now prove that this number N satisfies the implication above.

If $x > N$ we have $x > 2M$ and $x > R$. Therefore,

$$f(x) = x \cdot \frac{1 - 3/x^2 - 2/x^3}{1 - 1/x - 2/x^2} > 2M \cdot \frac{1}{2} = M.$$

This proves that $f(x) > M$ whenever $x > N$ and proves that

$$\lim_{x\to\infty} f(x) = \infty.$$

4. A rectangular billboard is to be made with 100 m^2 of printed area and with margins of 2 m at the top and bottom and 4 m on each side. Find the outside dimensions of the billboard if its total area is to be a minimum.

Solution. Let H denote the total height of the billboard and W its total width (both in meters). Since we require margins of 2 m at the top and bottom and 4 m on both sides, the height of the printed area is $(H-4)$ m and width of the printed area is $(W-8)$ m. Observe that the height and width of the printed area must be positive numbers, which means that we must have $H > 4$ and $W > 8$.

Since the printed area is 100 m^2 we must have

$$(H-4)(W-8) = 100$$

The equality above gives

$$H = 4 + \frac{100}{W-8}.$$

Observe that we do not have to worry about dividing by zero since $W > 8$.

Let A denote the total area of the bilboard. Then $A = HW$ and we have

$$A = HW = \left(4 + \frac{100}{W-8}\right) W.$$

Our task is to find which value of W gives the smallest area A, that is, we wish to minimize the function

$$f(W) = \left(4 + \frac{100}{W-8}\right) W$$

on the interval $(8, \infty)$.

The function f does not have any singular points on the interval $(8, \infty)$. Let us find its critical points (points where the derivative of f is equal to zero). We have

$$\begin{aligned} f'(W) &= -\frac{100}{(W-8)^2} W + 4 + \frac{100}{W-8} \\ &= -\frac{100W}{(W-8)^2} + \frac{4(W-8)^2}{(W-8^2)} + \frac{100(W-8)}{(W-8)^2} \\ &= \frac{-100W + 4(W-8)^2 + 100(W-8)}{(W-8)^2}. \end{aligned}$$

Thus, $f'(W) = 0$ if and only if

$$-100W + 4(W-8)^2 + 100(W-8) = 0.$$

Observe that

$$\begin{aligned} &-100W + 4(W-8)^2 + 100(W-8) \\ &\quad = -100W + 4(W-8)^2 + 100W - 800 \\ &\quad = 4(W-8)^2 - 800 \end{aligned}$$

Therefore, we get

$$\begin{aligned} f'(W) = 0 \quad &\iff \quad -100W + 4(W-8)^2 + 100(W-8) = 0 \\ &\iff \quad 4(W-8)^2 - 800 = 0 \\ &\iff \quad 4(W-8)^2 = 800 \\ &\iff \quad (W-8)^2 = 200 \\ &\iff \quad W - 8 = \pm\sqrt{200} \\ &\iff \quad W = 8 \pm \sqrt{200} \end{aligned}$$

Recall that we require $W > 8$, therefore we conclude that $f'(W) = 0$ if and only if $W = 8 + \sqrt{200}$.

This means that $f'(W) = 0$ if and only if $4(W-8)^2 = 800$. This means that there is only one critical point and it is

$$W = 8 + \sqrt{200}.$$

Since

$$\lim_{W \to 8+} f(W) = \infty \qquad \text{and} \qquad \lim_{W \to \infty} f(W) = \infty$$

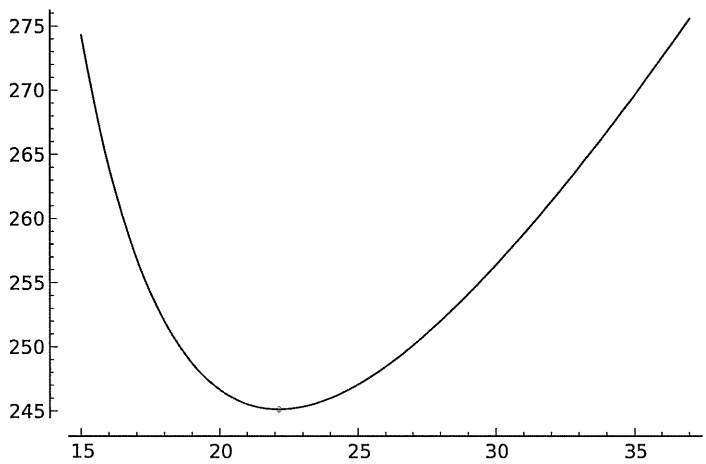
we are assured that the function f attains its minimum at the critical point.

We have showed that the total area A of the billboard is minimal for $W = 8 + \sqrt{200}$. Thus, we conclude that

$$\begin{aligned} W &= 8 + \sqrt{200} \approx 22.1 \text{ m} \\ H &= 4 + \frac{100}{W-8} = 4 + \frac{100}{\sqrt{200}} = 4 + \sqrt{50} \approx 11.1 \text{ m} \end{aligned}$$

and

$$A = HW = (4 + \sqrt{50})(8 + \sqrt{200}) \approx 245 \text{ m}^2.$$



5. Let $a < b$. Show that the derivative of $f(x) = (x-a)^m(x-b)^n$ vanishes at some point c in the interval (a, b) if m and n are positive integers.

Solution. Below you will find two solutions of this problem, the first based on using Rolle's Theorem and the second relying only on straight-forward calculations. The advantage of the second solution is that we find the value of c.

1) Recall Rolle's Theorem: Suppose that the function g is continuous on the bounded interval $[a, b]$ and that it is differentiable on the open interval (a, b). If $g(a) = g(b)$, then there exists a point c in the interval (a, b) such that $g'(c) = 0$.

Observe that the function $f(x) = (x-a)^m(x-b)^n$ is continuous on the interval $[a, b]$ and it is differentiable on (a, b). Moreover, we have $f(a) = f(b) = 0$. According to Rolle's Theorem, there exists a point c in the interval (a, b) such that $f'(c) = 0$. This finishes the proof.

2) Let us first calculate the derivative of f. We get

$$\begin{aligned} f'(x) &= m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1} \\ &= (x-a)^{m-1}(x-b)^{n-1}\big(m(x-b) + n(x-a)\big), \end{aligned}$$

where $m - 1 \geq 0$ and $n - 1 \geq 0$. We wish to find c such that $a < c < b$ and $f'(c) = 0$. By the above, $f'(c) = 0$ for c in (a, b) if and only if

$$m(c-b) + n(c-a) = 0.$$

Next, we determine the value of c as follows: First we observe that

$$0 = m(c-b) + n(c-a) = mc - bm + nc - an = (m+n)c - an - bm$$

which means that we must have

$$(m+n)c = an + bm.$$

Since m and n are positive integers we have $m+n > 0$. Thus, the number

$$c = \frac{an + bm}{m+n}$$

is well-defined. Using the fact that $a < b$ we conclude that

$$c = \frac{an + bm}{m+n} > \frac{an + am}{m+n} = a$$

and

$$c = \frac{an + bm}{m+n} < \frac{bn + bm}{m+n} = b$$

This shows that there exists a number c such that $a < c < b$ and $f'(c) = 0$.

6. Consider the function $f(x) = \sin x$.

(a) Find the general form of Taylor's formula for f at the point $a = 0$ with Lagrange remainder.

(b) Find a sufficiently large order n for which the corresponding Taylor polynomial approximation will give the sine of 1 radian correct to 5 decimal places. Approximate $\sin 1$ by using the Taylor polynomial of this order.

Solution. Recall Taylor's Theorem (with $a = 0$): for each non-negative integer n there is a real number c between x and 0 such that

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n + \frac{f^{(n+1)}(c)}{(n+1)!}\,x^{n+1},$$

where the last term on the right-hand side is the Lagrange remainder. Taking derivatives of $f(x) = \sin x$ at the point $x = 0$, we get

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1
6	$-\sin x$	0
7	$-\cos x$	-1
8	$\sin x$	0
$\vdots$	$\vdots$	$\vdots$

Looking at the value of $f^{(n)}(0)$ we see that it is zero if n is an even integer and ± 1 for odd n. If n is odd it may be expressed as $n = 2m+1$, where m is an integer. Observe that $f^{(2m-1)}(0) = (-1)^m$ for positive integers m. We conclude that for non-negative integers n, which corresponds to $n = 2m$ (if n is even) and $n = 2m-1$ (if n is odd) for non-negative integers m, we have

$$f^{(n)}(0) = \begin{cases} 0 & n = 2m, \\ (-1)^m & n = 2m+1. \end{cases}$$

Thus, if n is an even non-negative integer and $n = 2m$ we have

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m-1}}{(2m-1)!} + (-1)^m \frac{\cos c}{(2m+1)!}\,x^{2m+1},$$

and for $n = 2m+1$ we have

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + (-1)^{(m+1)} \frac{\sin c}{(2m+2)!}\,x^{2m+2}.$$

b) It is important to observe that the Taylor expansions of f about zero above only differs in the remainder for $n = 2m + 2$ and $n = 2m + 1$. Indeed, with order $n = 2m + 2$ we have

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{(m+1)}\frac{x^{2m+1}}{(2m+1)!} + (-1)^{(m+1)}\frac{\cos c}{(2m+3)!}x^{2m+3}.$$

For $x = 1$ we get the same Taylor polynomial with order $n = 2m+1$ and $n = 2m+2$, but the remainder is guaranteed to be smaller as $n = 2m+2$. How small must the order n be in order to guarantee correct decimals to five places? We need to assure that the remainder is smaller than $0.5 \cdot 10^{-5}$.

Observe that, for order $n = 2m + 2$, we have control of the size of the remained in the Taylor expansion as follows:

$$\left|(-1)^{m+1}\frac{\cos c}{(2m+3)!}\right| < \frac{1}{(2m+3)!}$$

It follows that it is enough to choose $n = 2m + 2$ big enough so that

$$\frac{1}{(2m+3)!} < 0.5 \cdot 10^{-5} \quad \Longleftrightarrow \quad 200\,000 < (2m+3)!$$

Observe that $8! = 40\,320$ and $9! = 362\,880$, which requires $m = 3$. This means that the Taylor polynomial of orden $n = 2m + 2 = 8$ is sufficient in order to guarantee an approximation of $\sin 1$ correct to five decimal places.

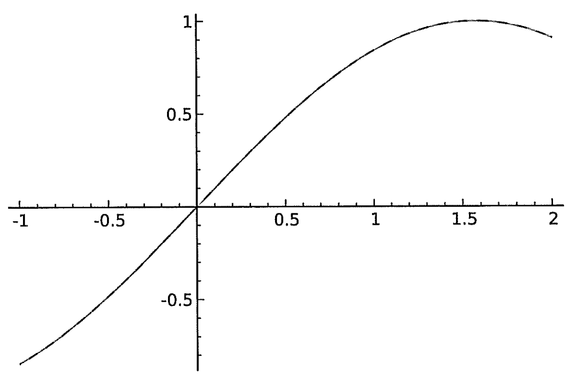
Using order $n = 8$ means that the polynomial

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

approximates $\sin 1$ good enough with $x = 1$. We get

$$\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} = 0.84147$$

using five decimal places.



7. (a) Solve the initial value problem

$$2y' = xy, \qquad y(0) = 1.$$

(b) Use Euler's Improved Method with $h = 1$ in order to approximate the value $y(2)$ of the solution to the equation above. Recall that the iterations used in Euler's Improved Method for approximating the solution the equation $y' = F(x, y)$, $y(x_0) = y_0$, are of the form

$$\begin{cases} x_{n+1} = x_n + h, \\ u_{n+1} = y_n + h \cdot F(x_n, y_n), \\ y_{n+1} = y_n + h \cdot \big(F(x_n, y_n) + F(x_{n+1}, u_{n+1})\big)/2. \end{cases}$$

Solution. a) This problem can be solved in at least two ways: 1) by use of the fact that the equation is separable; 2) using an integrating factor.

1) The equation may be written as

$$\frac{y'}{y} = \frac{x}{2}.$$

Integrating both sides with respect to x, we get

$$\ln|y| = \int \frac{y'}{y}\,dx = \int \frac{x}{2}\,dx = \frac{x^2}{4} + C, \qquad C \in \mathbb{R}.$$

Applying the exponential function we get

$$|y| = e^{\ln|y|} = e^{\frac{x^2}{4}+C} = e^C e^{x^2/4} = De^{x^2/4}, \qquad D = e^C \geq 0.$$

We conclude that all solutions to the equation $2y' = xy$ are given by $y(x) = De^{x^2/4}$, where D is any real number. The initial condition $y(0) = 1$ gives

$$1 = y(0) = De^0 = D.$$

We conclude that the solution is given by $y(x) = e^{x^2/4}$.

2) The equation may be written as

$$y' - \frac{x}{2}y = 0.$$

Thus, the integrating factor is given by

$$e^{\int -x/2\,dx} = e^{-x^2/4}.$$

Multiplying both sides of the equation by the integrating factor, the equation may be written as

$$\frac{d}{dx}\left(e^{-x^2/4}y\right) = e^{-x^2/4}\left(y' - \frac{x}{2}y\right) = 0.$$

Next, we integrate both sides of this equation with respect to x in order to get

$$e^{-x^2/4}y = \int \frac{d}{dx}\left(e^{-x^2/4}y\right)dx = \int 0\,dx = C, \qquad C \in \mathbb{R},$$

and we conclude that $y(x) = Ce^{x^2/4}$, $C \in \mathbb{R}$. Using the initial value $y(0) = 1$, we get

$$1 = y(0) = Ce^0 = C.$$

We conclude that the solution is given by $y(x) = e^{x^2/4}$.

b) The equation may be written as

$$y' = \frac{xy}{2} = F(x, y),$$

with initial values $x_0 = 0$ and $y_0 = 1$. With $h = 1$ we get

$$\begin{cases} x_1 = x_0 + h = 0 + 1 = 1, \\ u_1 = y_0 + h \cdot F(x_0, y_0) = 1 + F(0, 1) = 1, \\ y_1 = y_0 + h \cdot \dfrac{F(x_0, y_0) + F(x_1, u_1)}{2} = 1 + \dfrac{0 + 1/2}{2} = \dfrac{5}{4}, \end{cases}$$

and

$$\begin{cases} x_2 = x_1 + h = 1 + 1 = 2, \\ u_2 = y_1 + h \cdot F(x_1, y_1) = \dfrac{5}{4} + F(1, 5/4) = \dfrac{15}{8}, \\ y_2 = y_1 + h \cdot \dfrac{F(x_1, y_1) + F(x_2, u_2)}{2} = \dfrac{5}{4} + \dfrac{5/8 + 15/8}{2} = \dfrac{40}{16} = \dfrac{5}{2}. \end{cases}$$

We conclude that the approximated value of $y(2)$ is $5/2 = 2.5$. In view of 7.a) we now that the exact value is given by $y(2) = e^1 \approx 2.7182$.

8. (a) Approximate the integral

$$\int_1^3 xe^{-x^2}\,dx$$

by use of the Trapezoid Rule with $h = 0.25$. Recall that the Trapezoid rule for n steps is given by

$$T_n = h \cdot \left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right),$$

where $x_j = x_0 + j \cdot h,\ \ 0 \le j \le n.$

(b) Is the integral

$$\int_1^\infty xe^{-x^2}\,dx$$

convergent or divergent? Find its value if it is convergent or explain why it diverges.

Solution. a) Let $f(x) = xe^{-x^2}$ and set $x_0 = 1,\ h = 0.25$ and

$$x_n = x_{n-1} + h = x_0 + nh \qquad n = 1, 2, 3, \ldots, 8.$$

Thus, we have

$$x_0 = 1, \quad x_1 = 1.25, \quad x_2 = 1.5, \quad x_3 = 1.75, \quad x_4 = 2,$$

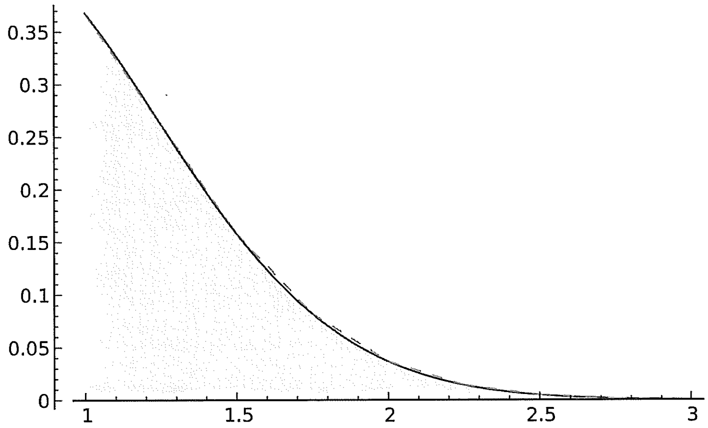
$$x_5 = 2.25, \quad x_6 = 2.5, \quad x_7 = 2.75, \quad \text{and} \quad x_8 = 3.$$

Calculating T_8, we get

$$\begin{aligned} T_8 &= h\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_7) + \frac{f(x_8)}{2} \right) \\ &= h\left(\frac{x_0 e^{x_0^2}}{2} + x_1 e^{x_1^2} + x_2 e^{x_2^2} + x_3 e^{x_3^2} \right. \\ &\qquad \left. + x_4 e^{x_4^2} + x_5 e^{x_5^2} + x_6 e^{x_6^2} + x_7 e^{x_7^2} + \frac{x_8 e^{x_8^2}}{2} \right) \\ &\approx 0.18580365 \end{aligned}$$

Observe that the exact value of the integral we are approximating is

$$\frac{1}{2}\left(e^{-1} - e^{-9}\right) \approx 0.18387802.$$



b) According to the definition of indefinite integrals, we have

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{R\to\infty}\int_1^R xe^{-x^2}\,dx.$$

Since

$$F(x) = -\frac{1}{2}\,e^{-x^2}$$

is such that

$$F'(x) = xe^{-x^2},$$

the Fundamental The of Calculus gives

$$\int_1^R xe^{-x^2}\,dx = F(R) - F(1) = \frac{1}{2}\left(e^{-1} - e^{-R^2}\right).$$

Observe that

$$\lim_{R\to\infty} e^{-R^2} = 0,$$

which implies that

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{R\to\infty}\int_1^R xe^{-x^2}\,dx = \lim_{R\to\infty}\frac{1}{2}\left(e^{-1} - e^{-R^2}\right) = \frac{1}{2e}.$$

We conclude that the integral is convergent and its value is $1/2e \approx 0.183940$.