Mittuniversitetet
NAT, Stefan Borell

Svenska

Tentamen i matematik
Fördjupningskurs i analys (MA092G/MA093G)
30 maj 2011
Skrivtid: 5 timmar
Hjälpmedel: Godkänd miniräknare samt bifogad formelsamling

Till alla uppgifter skall fullständiga lösningar lämnas. Resonemang, ekvationslösningar och uträkningar får inte vara så knapphändiga att de blir svåra att följa. En uppgift per blad, skriv endast på en sida.

Betyg sätts efter hur väl lärandemålen är uppfyllda. Riktvärde för betygen är: A 22p, B 18p, C 14p, D 10p, E 9p. Aspektuppgiften, markerad A, kan höja betyget om den utförs väl med god motivering.

1. (a) Ge den formella definitionen av påståendet "Funktionen $f(x)$ är kontinuerlig i punkten $x = a$." (1p)

   (b) Bevisa att funktionen $f(x) = 2x - 6$ är kontinuerlig i punkten $x = \pi$ enbart med hjälp av den formella definitionen av gränsvärden. (2p)

2. Bestäm Taylorpolynomet av grad 2 till $\sin(e^x - 1)$ i punkten $x = 0$ och använd detta för att beräkna
$$\lim_{x\to 0} \frac{x - \sin(e^x - 1)}{x^2}. \qquad (3p)$$

3. Är serien
$$\frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \frac{1}{3^5} + \frac{2}{3^6} + \cdots$$
konvergent eller divergent? Beräkna värdet, om den är konvergent. (3p)

4. En rektangulär pool är 10 meter bred och 25 meter lång. Dess grunda del är längs med ena kortsidan. Vidare ges vattendjupet på x meters avstånd från den grunda sidan av $d(x) = \frac{x^2}{125} + 1$. Vad är volymen av vatten i poolen när den är full? (3p)

5. Är integralen
$$\int_0^1 x^2 \ln x \, dx$$
konvergent eller divergent? Beräkna värdet, om den är konvergent. (3p)

6. Lös initialvärdesproblemet
$$y' + y = e^x, \qquad y(0) = 1. \qquad (3p)$$

7. Betrakta integralen

$$\int_0^1 e^{-x^2}\,dx.$$

Det är välkänt att funktionen $f(x) = e^{-x^2}$ inte har primitiva funktioner som kan uttryckas via ändliga kombinationer av elementarfunktioner. Det innebär att det inte finns en explicit funktion $F(x)$ sådan att $F'(x) = f(x)$ som vi kan använda tillsammans med Integralkalkylens huvudsats.

Antag att $a < b$ och att n är ett positivt heltal. Vi delar upp intervallet $[a, b]$ i n delintervall av samma längd $h = (b-a)/n$. Låt $m_1, \ldots, m_n$ vara mittpunkterna till dessa delintervall. För en funktion g på intervallet $[a, b]$ så ges Mittpunktsmetoden av $M_n = h\big(g(m_1) + \cdots + g(m_n)\big)$.

(a) Approximera värdet på integralen ovan genom att använda Mittpunktsmetoden för en uppdelning av intervallet $[0, 1]$ i åtta delintervall. (2p)

(b) Antag att g har en kontinuerlig andraderivata på $[a, b]$. Då ges feluppskattningen av Mittpunktsmetoden av

$$\left|\int_a^b g(x)\,dx - M_n\right| \leq K\,\frac{(b-a)}{24}\,h^2,$$

där $K > 0$ är ett tal sådant att $|g''(x)| \leq K$ för alla x i $[0, 1]$.

Gör en feluppskattning för approximeringen som beräknades i 7.(a) genom att använda det faktum att $|f''(x)| \leq 2$ för x i $[0, 1]$. Hur många korrekta decimaler ger approximeringen i 7.(a)? (1p)

8. Är följande påståenden SANNA eller FALSKA? Ingen motivering krävs. Ett korrekt svar ger 1/2 poäng och ett felaktigt svar ger $-1/2$ poäng. Poängen för den här uppgiften ges av det minsta icke-negativa heltalet som är större än eller lika med det totala antalet poäng som införskaffats genom att svara på 8.(a)–(e).

(a) Om $\lim_{x\to a} f(x)$ existerar och $\lim_{x\to a} g(x)$ inte existerar, så existerar inte gränsvärdet $\lim_{x\to a}\big(f(x) + g(x)\big)$.

(b) Om varken $\lim_{x\to a} f(x)$ eller $\lim_{x\to a} g(x)$ existerar, så existerar inte gränsvärdet $\lim_{x\to a}\big(f(x) + g(x)\big)$.

(c) Om f är kontinuerlig i a, så är $|f|$ kontinuerlig i a.

(d) Om $|f|$ är kontinuerlig i a, så är f kontinuerlig i a.

(e) Om $f(x) < g(x)$ för alla x i ett intervall kring a, och om $\lim_{x\to a} f(x)$ och $\lim_{x\to a} g(x)$ existerar, så är $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

(3p)

Ej!

9. Antag att f är kontinuerlig på $[a, b]$, $a < b$. Bestäm konstanten k som minimerar integralen $\int_a^b \big(f(x) - k\big)^2\,dx$.

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Engelska

Mathematics Examination
Calculus II (MA092G/MA093G)
30 May 2011
Duration: 5 hours
Aids permitted: An approved calculator and the formula collection attached

Working must be shown in full in order to obtain full credit for an exercise. Do one question per page and write on one side of the paper only.

The grade is determined by the extent to which a candidate demonstrates that the learning outcomes of the course have been met. Guide values for grades are: A 22p, B 18p, C 14p, D 10p, E 9p. The Aspect Question, denoted A, may raise the final grade by one if it is solved in full with detailed motivation.

1. (a) Give the formal definition of the statement "The function $f(x)$ is continuous at the point $x = a$." (1p)

   (b) Using only the formal definition of limits, prove that the function $f(x) = 2x - 6$ is continuous at the point $x = \pi$. (2p)

2. Find the Taylor polynomial of degree 2 for $\sin(e^x - 1)$ about $x = 0$ and use it to calculate
$$\lim_{x \to 0} \frac{x - \sin(e^x - 1)}{x^2}.$$ (3p)

3. Is the series
$$\frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \frac{1}{3^5} + \frac{2}{3^6} + \cdots$$
convergent or divergent? If it is convergent, calculate its value. (3p)

4. A rectangular pool is 10 meters wide and 25 meters long. Its shallow end is along one of the 10 meter sides. The depth $d(x)$ of water at x meter from the shallow end of the pool is given by $d(x) = \frac{x^2}{125} + 1$. What is the exact volume of water in the pool when it is completely filled with water? (3p)

5. Is the integral
$$\int_0^1 x^2 \ln x \, dx$$
convergent or divergent? If it is convergent, calculate its value. (3p)

6. Solve the initial value problem
$$y' + y = e^x, \qquad y(0) = 1.$$ (3p)

7. Consider the integral

$$\int_0^1 e^{-x^2}\,dx.$$

It is known that the function $f(x) = e^{-x^2}$ does not possess antiderivatives that can be expressed as finite combinations of elementary functions. Therefore, since we cannot find an explicit function $F(x)$ such that $F'(x) = e^{-x^2}$, we cannot calculate this integral using the Fundamental Theorem of Calculus.

Suppose that $a < b$ and let n be a positive integer. We split the interval $[a, b]$ into n subintervals of equal length $h = (b-a)/n$. Let $m_1, \ldots, m_n$ be the midpoints of these subintervals. For a function c on the interval $[a, b]$, the Midpoint Rule is given by $M_n = h\big(g(m_1) + \cdots + g(m_n)\big)$.

(a) Find an approximate value of the integral above using the Midpoint Rule by dividing the interval $[0, 1]$ into eight subintervals. (2p)

(b) Suppose that g has a continuous second derivative on $[a, b]$. Then error estimates for the Midpoint Rule are given by

$$\left|\int_a^b g(x)\,dx - M_n\right| \le K\,\frac{(b-a)}{24}\,h^2,$$

where $K > 0$ is such that $|g''(x)| \le K$ for all x in $[0, 1]$.

Find an estimate of the error in the approximation calculated in 7.(a) using the fact that $|f''(x)| \le 2$ for x in $[0, 1]$. How many correct digits do the approximation in 7.(a) give? (1p)

8. Are the following statements TRUE or FALSE? You do not have to motivate your answers. A correct answer gives 1/2 a point and an incorrect answer gives $-1/2$ a point. The total score is given by the smallest non-negative integer which is greater than or equal to the total number of points gathered from answering 8.(a)–(e).

(a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then the limit $\lim_{x\to a}\big(f(x) + g(x)\big)$ does not exist.

(b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then the limit $\lim_{x\to a}\big(f(x) + g(x)\big)$ does not exist.

(c) If f is continuous at a, then $|f|$ is continuous at a.

(d) If $|f|$ is continuous at a, then f is continuous at a.

(e) If $f(x) < g(x)$ for all x in an interval around a, and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

(3p)

Ej! ~~9.~~ Suppose that f is continuous on $[a, b]$, $a < b$. Find the constant k that minimises the integral $\displaystyle\int_a^b \big(f(x) - k\big)^2\,dx$.

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"Solutions" of the Exam Problems
(Some details may have been left out)

1. (a) Give the formal definition of the statement "The function $f(x)$ is continuous at the point $x = a$."

   (b) Using only the formal definition of limits, prove that the function $f(x) = 2x - 6$ is continuous at the point $x = \pi$.

   Solution. a) The statement is defined as follows:

$$\lim_{x \to a} f(x) = f(a).$$

b) We wish to confirm that

$$\lim_{x \to \pi} f(x) = f(\pi) = 2\pi - 6.$$

According to the formal definition of a limit, this is equivalent to the following: For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - \pi| < \delta \quad \Longrightarrow \quad |f(x) - (2\pi - 6)| < \varepsilon.$$

Since $f(x) = 2x - 6$, we have

$$|f(x) - (2\pi - 6)| = |2x - 6 - (2\pi - 6)| = |2x - 2\pi| = 2|x - \pi|.$$

Therefore, for $0 < |x - \pi| < \delta$, we get

$$|f(x) - (2\pi - 6)| = 2|x - \pi| < 2\delta,$$

and this assures that $|f(x) - (2\pi - 6)| < \varepsilon$ if $2\delta \leq \varepsilon$. Choosing $\delta = \varepsilon/2$ we are guaranteed that

$$0 < |x - \pi| < \delta = \varepsilon/2 \quad \Longrightarrow \quad |f(x) - (2\pi - 6)| < 2\delta = \varepsilon.$$

2. Find the Taylor polynomial of degree 2 for $\sin(e^x - 1)$ about $x = 0$ and use it to calculate

$$\lim_{x\to 0} \frac{x - \sin(e^x - 1)}{x^2}.$$

Solution. Let $f(x) = \sin(e^x - 1)$. Then $f(0) = 0$,

$$f'(x) = e^x \cos(e^x - 1), \qquad f'(0) = 1,$$

and

$$f''(x) = e^x \cos(e^x - 1) - e^{2x} \sin(e^x - 1), \qquad f''(0) = 1.$$

Thus,

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + O(x^3) = x + x^2/2 + O(x^3)$$

as x approaches 0. Using this fact, we get

$$\lim_{x\to 0} \frac{x - \sin(e^x - 1)}{x^2} = \lim_{x\to 0} \frac{x - x - x^2/2 - O(x^3)}{x^2} = \lim_{x\to 0} \frac{-x^2/2 - O(x^3)}{x^2}.$$

Since

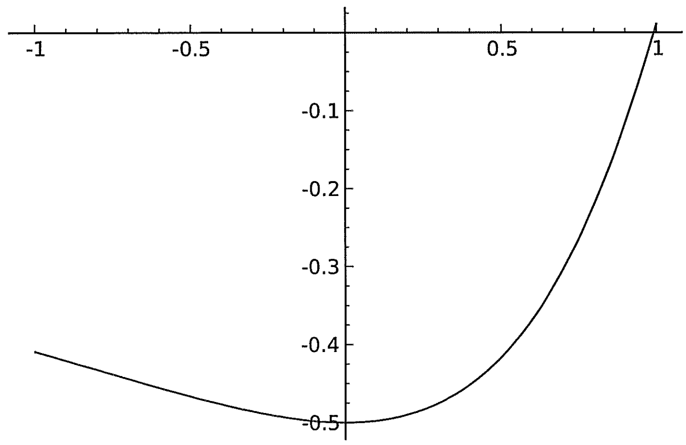
$$\lim_{x\to 0} \frac{-x^2/2}{x^2} = \lim_{x\to 0} -\frac{1}{2} = -\frac{1}{2}$$

and

$$\lim_{x\to 0} \frac{O(x^3)}{x^2} = 0,$$

the limit rule for subtractions give

$$\begin{aligned} \lim_{x\to 0} \frac{x - \sin(e^x - 1)}{x^2} &= \lim_{x\to 0} \frac{-x^2/2 - O(x^3)}{x^2} \\ &= \lim_{x\to 0} \frac{-x^2/2}{x^2} - \lim_{x\to 0} \frac{O(x^3)}{x^2} = -\frac{1}{2} - 0 = -\frac{1}{2}. \end{aligned}$$



3. Is the series
$$\frac{1}{3}+\frac{2}{3^2}+\frac{1}{3^3}+\frac{2}{3^4}+\frac{1}{3^5}+\frac{2}{3^6}+\cdots$$
convergent or divergent? If it is convergent, calculate its value.

Solution. Let
$$a_n=\frac{1}{3^{2n-1}}+\frac{2}{3^{2n}},$$
then
$$\frac{1}{3}+\frac{2}{3^2}+\frac{1}{3^3}+\frac{2}{3^4}+\frac{1}{3^5}+\frac{2}{3^6}+\cdots=\sum_{n=1}^{\infty}a_n.$$
Thus, for any positive integer N we have
$$\begin{aligned}s_N=\sum_{n=1}^{N}a_n&=\sum_{n=1}^{N}\frac{1}{3^{2n-1}}+\sum_{n=1}^{N}\frac{2}{3^{2n}}\\&=3\cdot\sum_{n=1}^{N}\frac{1}{9^n}+2\cdot\sum_{n=1}^{N}\frac{1}{9^n}=5\cdot\sum_{n=1}^{N}\frac{1}{9^n}=\frac{5}{9}\cdot\sum_{n=0}^{N-1}\frac{1}{9^n}\end{aligned}$$
Considering geometric series we get
$$\lim_{N\to\infty}\sum_{n=0}^{N-1}\frac{1}{9^n}=\sum_{n=0}^{\infty}\frac{1}{9^n}=\frac{1}{1-1/9}=\frac{9}{8}$$
and therefore, using the limit rule for scaling we have
$$\sum_{n=1}^{\infty}a_n=\lim_{N\to\infty}s_N=\lim_{N\to\infty}\frac{5}{9}\cdot\sum_{n=0}^{N-1}\frac{1}{9^n}=\frac{5}{9}\cdot\lim_{N\to\infty}\sum_{n=0}^{N-1}\frac{1}{9^n}=\frac{5}{9}\cdot\frac{9}{8}=\frac{5}{8}.$$
This shows that the series is convergent and its value is $5/8$.

4. A rectangular pool is 10 meters wide and 25 meters long. Its shallow end is along one of the 10 meter sides. The depth $d(x)$ of water at x meter from the shallow end of the pool is given by $d(x) = \frac{x^2}{125} + 1$. What is the exact volume of water in the pool when it is completely filled with water?

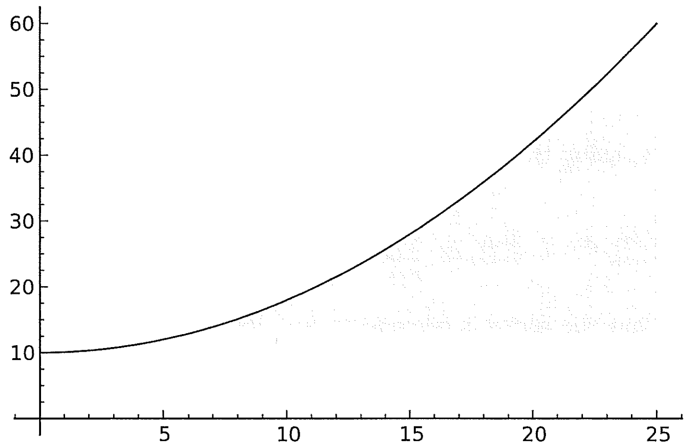
Solution. The volume of the pool is given by the integral

$$\int_0^{25} 10 \cdot d(x)\, dx.$$

Calculating this integral, we get

$$\begin{aligned} 10 \int_0^{25} (x^2/125 + 1)\, dx &= 10 \left[\frac{x^3}{3 \cdot 125} + x \right]_0^{25} \\ &= 10 \left(\frac{25^3}{3 \cdot 125} + 25 \right) = 10 \cdot \frac{200}{3} = \frac{2000}{3}. \end{aligned}$$

Since $2000/3 \approx 670$ and the unit is m^3, this means that there is approximately 670 000 litres of water in the pool.



5. Is the integral

$$\int_0^1 x^2 \ln x \, dx$$

convergent or divergent? If it is convergent, calculate its value.

Solution. Since the function $f(x) = x^2 \ln x$ is continuous on $(0,1]$, we need to check whether the limit

$$\lim_{c \to 0+} \int_c^1 x^2 \ln x \, dx$$

exists, or not. By partial integration we get

$$\int_c^1 x^2 \ln x \, dx = \left[\frac{x^3}{3} \cdot \ln x \right]_c^1 - \int_c^1 \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

$$= \left[\frac{x^3}{3} \cdot \ln x \right]_c^1 - \left[\frac{x^3}{9} \right]_c^1 = 0 - \frac{c^3}{3} \cdot \ln c - \frac{1}{9} + \frac{c^3}{9}$$

Using the limits

$$\lim_{c \to 0+} -\frac{c^2}{3} = 0, \qquad \lim_{c \to 0+} c \cdot \ln c = 0,$$

$$\lim_{c \to 0+} -\frac{1}{9} = -\frac{1}{9}, \qquad \lim_{c \to 0+} \frac{c^3}{9} = 0,$$

together with the limit rules for addition and multiplication we get

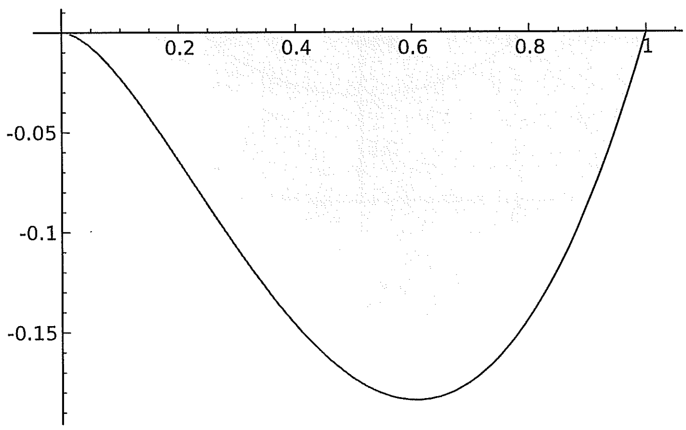
$$\lim_{c \to 0+} \int_c^1 x^2 \ln x \, dx = \lim_{c \to 0+} \left(-\frac{c^3}{9} \cdot \ln c - \frac{1}{9} + \frac{c^3}{9} \right)$$

$$= \left(\lim_{c \to 0+} -\frac{c^2}{3} \right) \left(\lim_{c \to 0+} c \ln c \right) - \lim_{c \to 0+} \frac{1}{9} + \lim_{c \to 0+} \frac{c^3}{9} = -\frac{1}{9}.$$

This shows that the integral

$$\int_0^1 x^2 \ln x \, dx$$

is convergent and its value is $-1/9$.



6. Solve the initial value problem

$$y' + y = e^x, \qquad y(0) = 1.$$

Solution. We solve this problem by use of integrating factors. Let $F(x) = x$, then $F'(x) = 1$ is equal to the coefficient of y. This gives us the integrating factor $e^{F(x)} = e^x$. Multiplying the original equality with this factor, we get

$$\frac{d}{dx}(e^x y) = e^x y' + e^x y = e^{2x}.$$

Next, we integrate both sides and get

$$e^x y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C, \qquad C \in \mathbb{R}.$$

We conclude that

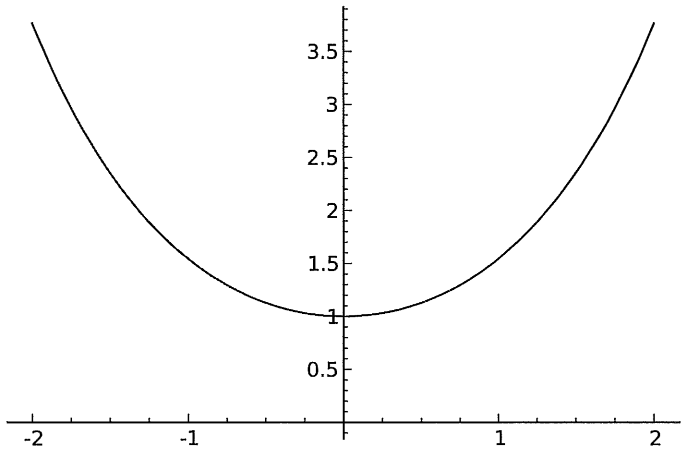
$$y(x) = \frac{1}{2}e^x + Ce^{-x}, \qquad C \in \mathbb{R}.$$

The function y satisfies the initial condition if and only if

$$1 = y(0) = \frac{1}{2} + C \quad \Longleftrightarrow \quad C = \frac{1}{2}.$$

The solution curve satisfying the initial value problem $y' + y = e^x$, $y(0) = 1$ is

$$y(x) = (e^x + e^{-x})/2.$$



7. Consider the integral

$$\int_0^1 e^{-x^2}\, dx.$$

It is known that the function $f(x) = e^{-x^2}$ does not possess antiderivatives that can be expressed as finite combinations of elementary functions. Therefore, since we cannot find an explicit function $F(x)$ such that $F'(x) = e^{-x^2}$, we cannot calculate this integral using the Fundamental Theorem of Calculus.

Suppose that $a < b$ and let n be a positive integer. We split the interval $[a, b]$ into n subintervals of equal length $h = (b-a)/n$. Let $m_1, \ldots, m_n$ be the midpoints of these subintervals. For a function c on the interval $[a, b]$, the Midpoint Rule is given by $M_n = h\big(g(m_1) + \cdots + g(m_n)\big)$.

(a) Find an approximate value of the integral above using the Midpoint Rule by dividing the interval $[0, 1]$ into eight subintervals.

(b) Suppose that g has a continuous second derivative on $[a, b]$. Then error estimates for the Midpoint Rule are given by

$$\left|\int_a^b g(x)\, dx - M_n\right| \leq K\, \frac{(b-a)}{24}\, h^2,$$

where $K > 0$ is such that $|g''(x)| \leq K$ for all x in $[0, 1]$.

Find an estimate of the error in the approximation calculated in 7.(a) using the fact that $|f''(x)| \leq 2$ for x in $[0, 1]$. How many correct digits do the approximation in 7.(a) give?

Solution. a) Dividing the interval $[0, 1]$ into eight parts of equal length $h = 1/8$ we get the following subintervals:

- $[0, 1/8]$ with midpoint $m_1 = 1/16$,
- $[1/8, 2/8]$ with midpoint $m_2 = 3/16$,
- $[2/8, 3/8]$ with midpoint $m_3 = 5/16$,
- $[3/8, 4/8]$ with midpoint $m_4 = 7/16$,
- $[4/8, 5/8]$ with midpoint $m_5 = 9/16$,
- $[5/8, 6/8]$ with midpoint $m_6 = 11/16$,
- $[6/8, 7/8]$ with midpoint $m_7 = 13/16$; and
- $[7/8, 1]$ with midpoint $m_8 = 15/16$.

Thus, using the Midpoint Rule we get the approximative value

$$\begin{aligned} M_8 &= \frac{1}{8}\left(e^{-m_1^2} + e^{-m_2^2} + e^{-m_3^2} + e^{-m_4^2} + e^{-m_5^2} + e^{-m_6^2} + e^{-m_7^2} + e^{-m_8^2}\right) \\ &\approx 0.747304 \end{aligned}$$

of the integral

$$\int_0^1 e^{-x^2}\, dx.$$

b) Using $a = 0$, $b = 1$, $h = 1/8$, and $K = 2$, the error estimates for the Midpoint Rule is

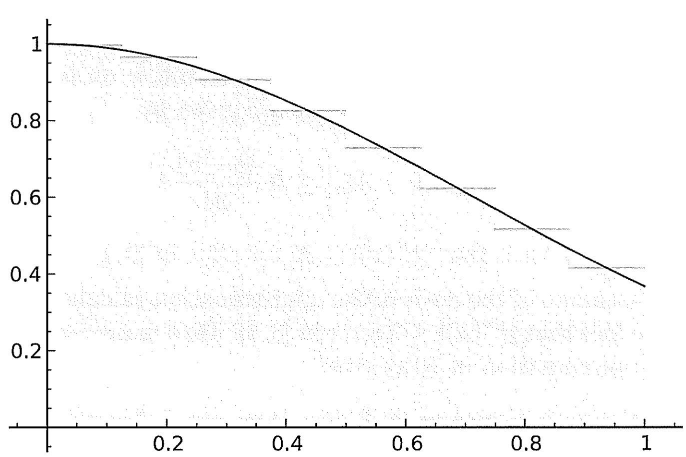
$$\left| \int_0^1 e^{-x^2}\,dx - M_8 \right| \leq K \frac{(b-a)}{24} h^2 = 2 \cdot \frac{1}{24} \cdot \frac{1}{8^2} = \frac{1}{768} \approx 0.00130.$$

This means that

$$\int_0^1 e^{-x^2}\,dx = 0.747304 \pm 0.00130$$

and we are assured that the correct value is between 0.746 and 0.748. Therefore we have two correct decimals in the approximation.

Note: The correct value to six decimal places is 0.746824.



8. Are the following statements TRUE or FALSE? You do not have to motivate your answers. A correct answer gives 1/2 a point and an incorrect answer gives $-1/2$ a point. The total score is given by the smallest non-negative integer which is greater than or equal to the total number of points gathered from answering 8.(a)–(e).

   (a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then the limit $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.

   (b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then the limit $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.

   (c) If f is continuous at a, then $|f|$ is continuous at a.

   (d) If $|f|$ is continuous at a, then f is continuous at a.

   (e) If $f(x) < g(x)$ for all x in an interval around a, and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

Solution.

(a) TRUE

(b) FALSE

(c) TRUE

(d) FALSE

(e) FALSE

A. Suppose that f is continuous on $[a, b]$, $a < b$. Find the constant k that minimises the integral $\int_a^b \big(f(x) - k\big)^2 dx$.

Solution. Since $f(x)$ is continuous on $[a, b]$, the same holds true for $\big(f(x)\big)^2$ and $\big(f(x) - k\big)^2$. Moreover,

$$\big(f(x) - k\big)^2 = \big(f(x)\big)^2 - 2kf(x) + k^2$$

and therefore

$$\int_a^b \big(f(x) - k\big)^2 \, dx = \int_a^b \big(f(x)\big)^2 dx - 2k \int_a^b f(x)\, dx + k^2 \int_a^b dx.$$

Set

$$A = \int_a^b \big(f(x)\big)^2 dx, \qquad B = \int_a^b f(x)\, dx,$$

and observe that

$$k^2 \int_a^b dx = (b - a)k^2.$$

Using this notation we have

$$\int_a^b \big(f(x) - k\big)^2 dx = A - 2Bk + (b - a)k^2.$$

Our task is to find the value of k that minimises the function

$$g(k) = A - 2Bk + (b - a)k^2.$$

Observe that $b > a$ gives

$$\lim_{k \to \pm\infty} g(k) = +\infty.$$

Since g is a polynomial in k there are no singular points to consider. Let us look for critical points. We get

$$g'(k) = -2B + 2(b - a)k = 0 \iff k = B/(b - a).$$

Thus, g must have a minimum at $k = B/(b - a)$. This shows that the value of k that minimises the integral is

$$k = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

Note that this value of k is the mean value of the function f on the interval $[a, b]$.