Mid Sweden University
NAT
Björn Ivarsson, Abtin Daghighi

Tentamen in mathematics.
Fördjupningskurs i analys / Calculus II (MA059G/MA060G).
31 May 2010
Time for tenta: 5 hours
Allowed calculators and collections of formulas: Nonsymbolic calculators and approved "gymnasieformelsamling".

You must provide complete solutions. Motivations and calculations should be comprehensible and easy to follow. One problem per sheet and write only on one side of the paper.

The following guidelines will be used to grade: A 22 p, B 18 p, C 14p, D 10p, E 9p. The grade will be set by how satisfactory you meet the "lärandemål".

Ok!

(1) (a) State the formal definition of $\lim_{x\to a} f(x) = L$. (1p)

ok!

(b) Use the formal definition to verify that

$$\lim_{x\to 1} 2x + 1 = 3.$$

(2p)

ok!

(2) Is the integral

$$\int_0^1 \frac{\ln x}{x}\, dx$$

convergent or divergent? If it is convergent calculate the integrals value. (3p)

ok!

(3) Let T be a triangle with corners in $(0,0)$, $(a,0)$ and $(0,b)$ having area S, that is

$$S = \frac{ab}{2}.$$

Draw a rectangle in the triangle with corners in $(0,0)$, $(x,0)$, $(0,y)$ and on the hypothenuse (x,y). What is the maximal area that the rectangle can have? (3p)

ok!

(4) Find all solutions of

$$y' - \frac{2}{x}y = x^2 e^x, \ x > 0.$$

(3p)

ok!

(5) Find the Taylor polynomial of degree 3 for $\ln(1-x)$ about $x = 0$ and use it to calculate

$$\lim_{x\to 0} \frac{2\ln(1-x) + 2x + x^2}{4x^3 + x^4}.$$

(3p)

ok!

(6) (a) Does

$$\sum_{n=1}^{\infty} \frac{2^n + n^5}{3^n}$$

converge or diverge? (1p)

ok!

(b) Does

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

converge or diverge? (2p)

ok!

(7) Let $y(x)$ be the solution to the initial value problem $y' = xe^y$ such that $y(0) = 0$. Use the Euler method with step size $h = 0.2$ to approximate $y(1)$. (3p)

ok!

(8) (a) (Do this problem if you are studying MA059G) Use the Mean-Value Theorem to show that $\ln x < x - 1$ for all $x > 1$. (3p)

ok!

(b) (Do this problem if you are studying MA060G) Approximate the root of $e^{-x} = x$ by doing three iterations of the Fixed Point Method, that is calculate x_3, if $x_0 = 0.5$. (3p)

Good luck!

1a) State the formal definition of

$$\lim_{x \to a} f(x) = L$$

b) Use the formal definition to verify that

$$\lim_{x \to 1} 2x+1 = 3$$

Solution: a) For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

b) Choose $\varepsilon > 0$ and put $\delta = \frac{\varepsilon}{2}$.

If $0 < |x-1| < \delta$ then

$$|f(x)-3| = |2x+1-3| = |2x-2| = 2|x-1| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

⊗

2/ Is the integral $\int_0^1 \frac{\ln x}{x}\,dx$ convergent or divergent? If it is convergent calculate the integrals value.

Solution: We need to check if

$$\lim_{N \to 0^+} \int_N^1 \frac{\ln x}{x}\,dx \text{ exists.}$$

First calculate primitive functions.

$$\int \frac{\ln x}{x}\,dx = \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right] = \int u\,du = \frac{u^2}{2} + C =$$

$$= \frac{(\ln x)^2}{2} + C$$

We calculate

$$\lim_{N\to 0^+} \int_N^1 \frac{\ln x}{x}\,dx = \lim_{N\to 0^+} \left[\frac{(\ln x)^2}{2} \right]_N^1 =$$

$$= \lim_{N\to 0^+} \frac{(\ln 1)^2}{2} - \frac{(\ln N)^2}{2} = -\lim_{N\to 0^+} \frac{(\ln N)^2}{2} = -\infty$$

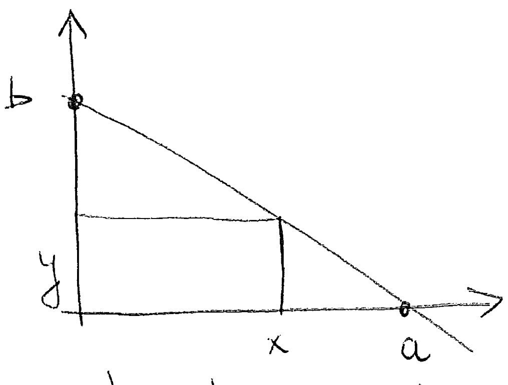
The integral $\int_0^1 \frac{\ln x}{x}\,dx$ is divergent.

3) Let T be a triangle with corners in $(0,0)$, $(a,0)$ and $(0,b)$ having area S, that is

$$S = \frac{ab}{2}.$$

Draw a rectangle in the triangle with corners $(0,0)$, $(x,0)$, $(0,y)$ and on the hypothenuse (x,y). What is the maximal area that the rectangle can have?

Solution: We have



The area of the rectangle is xy. Since (x,y) lies on the hypothenuse we have

$$y = -\frac{b}{a}x + b$$

So we need to maximize the function

$$A(x) = x \cdot y = x \cdot \left(-\frac{b}{a}x + b\right) = \underbrace{-\frac{b}{a}x^2 + bx}$$ when $0 \le x \le a$.

1) Critical points $A'(x) = 0$

$$A'(x) = -\frac{2b}{a}x + b = 0$$

$$\frac{2b}{a}x = b$$

$$x = \frac{a}{2}$$

2) Singular points does not exist since $A(x)$ is a polynomial

3) Endpoints $x = 0$ & $x = a$

4) Calculate and compare

$$A(0) = 0$$

$$A\left(\frac{a}{2}\right) = \frac{a}{2}\left(-\frac{b}{\cancel{a}} \cdot \frac{\cancel{a}}{2} + b\right) = \frac{a}{2} \cdot \frac{b}{2} = \frac{ab}{4}$$

$$A(a) = 0$$

Hence the maximal area the regtangle can have is $\frac{ab}{4} = \frac{S}{2}$ $\left(\text{since } S = \frac{ab}{2}\right)$

4) Find all solutions of

$$y' - \frac{2}{x}y = x^2e^x \quad , x>0$$

Solution: We use integrating factors.

$$e^{\int -\frac{2}{x}dx} = e^{-2\ln x\,(+C)} = x^{-2}$$

Multiply the equation with the integrating factor

$$x^{-2}y' - 2x^{-3}y = e^x$$

$$\frac{d}{dx}(x^{-2}y) = e^x$$

Integrate both sides

$$x^{-2}y = e^x + D$$

$$y = x^2e^x + Dx^2$$

5) Find the Taylor polynomial of degree 3 for $\ln(1-x)$ about $x=0$ and use it to calculate

$$\lim_{x\to 0} \frac{2\ln(1-x) + 2x + x^2}{4x^3 + x^4}$$

Solution: Let $f(x) = \ln(1-x)$. We find

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3.$$

$$f'(x) = -\frac{1}{1-x} = -(1-x)^{-1}$$

$$f''(x) = -(1-x)^{-2}$$

$$f^{(3)}(x) = -2(1-x)^{-3}$$

$$f(0) = \ln 1 = 0$$

$$f'(0) = -1$$

$$f''(0) = -1$$

$$f^{(3)}(0) = -2$$

$$T_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

We know that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + O(x^4)$$

and use this to calculate

$$\lim_{x\to 0} \frac{2\ln(1-x) + 2x + x^2}{4x^3 + x^4} = \lim_{x\to 0} \frac{-2x - x^2 - \frac{2}{3}x^3 + 2x + x^2 + O(x^4)}{4x^3 + x^4}$$

$$= \lim_{x\to 0} \frac{-\frac{2}{3}x^3 + O(x^4)}{4x^3 + x^4} = \lim_{x\to 0} \frac{-\frac{2}{3} + \frac{O(x^4)}{x^3}}{4 + x} = -\frac{2}{3\cdot 4} = -\frac{1}{6}$$

$\otimes$

6) a) Does $\sum_{n=1}^{\infty} \frac{2^n + n^5}{3^n}$ converge or diverge?

b) Does $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ converge or diverge?

Solution: a) We put $a_n = \frac{2^n + n^5}{3^n}$ and $b_n = \frac{2^n}{3^n}$.

We know that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converges since $\frac{2}{3} < 1$.

We use $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n+n^5}{3^n} \cdot \frac{3^n}{2^n} =$

$= \lim_{n\to\infty} 1 + \frac{n^5}{2^n} = 1$

to conclude that also

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2^n+n^5}{3^n}$ converges.

b) Since $\ln n \leq n$ when $n \geq 2$

we have $\frac{1}{n} \leq \frac{1}{\ln n}$ when $n \geq 2$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges.

7) Let $y(x)$ be the solution to the initial value problem $y' = xe^y$ such that $y(0)=0$. Use the Euler method with step size $h=0.2$ to approximate $y(1)$.

Solution: Remember

$x_{n+1} = x_n + h$ & $y_{n+1} = y_n + f(x_n, y_n)h$

where $y' = f(x,y) = xe^y$

The initial values are

$x_0 = 0 \qquad y_0 = 0$

$x_1 = 0{,}2 \qquad y_1 = 0 + x_0 e^{y_0} \cdot 0{,}2 = 0$

$x_2 = 0{,}4 \qquad y_2 = 0 + 0{,}2\, e^{0} \cdot 0{,}2 = 0{,}04$

$x_3 = 0{,}6 \qquad y_3 = 0{,}04 + 0{,}4 \cdot e^{0{,}04} \cdot 0{,}2 = 0{,}123264\ldots$

$x_4 = 0{,}8 \qquad y_4 = y_3 + 0{,}6\, e^{y_3} \cdot 0{,}2 = 0{,}259006\ldots$

$x_5 = 1 \qquad y_5 = y_4 + 0{,}8\, e^{y_4} \cdot 0{,}2 = 0{,}466309\ldots$

So $y(1) \approx 0{,}466309$

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8a) (If you are studying MA059G)
Use the Mean-Value Theorem to show that $\ln x < x - 1$ for all $x > 1$.

b) (If you are studying MA060G)
Approximate the root of $e^{-x} = x$ by doing 3 iterations of the Fixed Point Method, that is calculate x_3, if $x_0 = 0{,}5$.

Solution: a) Rewrite $\ln x < x - 1$ as

$$0 < x - 1 - \ln x = f(x) \quad \text{when } x > 1.$$

So we want to show that $f(x) > 0$ when $x > 1$

Note that $f(1) = 1 - 1 - \ln 1 = 0$

Assume there is an $x > 1$ such that
$f(x) = 0$.

By the Mean-Value Theorem there is a $1 < \xi < x$ such that $f'(\xi) = \dfrac{f(x) - f(1)}{x-1} = 0$

But $f'(x) = 1 - \frac{1}{x}$ and

$$1 - \frac{1}{\xi} = 0 \text{ iff } \xi = 1$$

so this is a contradiction

Therefore $f(x) > 0$ for $x > 1$ or

$f(x) < 0$ for $x > 1$.

We check which it is by evaluating:

$$f(e) = e - 1 - \ln e = e - 1 - 1 > 0$$

So $f(x) > 0$ for all $x > 1$

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b) The Fixed Point Method can be used to approximate roots of $x = f(x)$.

Here $f(x) = e^{-x}$

In general $x_{n+1} = f(x_n)$

Therefore

$$x_1 = e^{-x_0} = e^{-0{,}5} = 0{,}606530\ldots$$

$$x_2 = e^{-x_1} = 0{,}545239\ldots$$

$$x_3 = e^{-x_2} = 0{,}579703\ldots$$

We approximate the root to $y = e^{-x}$ with $0{,}579703\ldots$