Mid Sweden University
NAT
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Testtenta in mathematics.
Fördjupningskurs i analys / Calculus II (MA059G/MA060G).
May 2010
Time for tenta: 5 hours
Allowed calculators and collections of formulas: Nonsymbolic calculators and approved "gymnasieformelsamling".

You must provide complete solutions. Motivations and calculations should be comprehensible and easy to follow. One problem per sheet and write only on one side of the paper.

The following guidelines will be used to grade: A 22 p, B 18 p, C 14p, D 10p, E 9p. The grade will be set by how satisfactory you meet the "lärandemål".

Ok!

(1) (a) Give the formal definition of the statement: "The function $f(x)$ is continuous at the point $x = a$". (1p)

Ok!

(b) Is the function defined as $f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ continuous at the point $x = 0$? (2p)

Ok!

(2) Is the integral

$$\int_1^{\infty} xe^{-x}\, dx$$

convergent or divergent? If it is convergent calculate the integrals value. (3p)

Ok!

(3) An orchard has 60 trees and produce an average of 800 apples per tree per year. If more trees are planted the yield per tree will drop. For each tree planted the average yield per tree is reduced by 10 apples per year. How many more trees should be planted in order to maximize the yield from the orchard and what is the maximal yield? (3p)

orchard = fruktodling

Ok!

(4) Find all solutions to

$$xy'' + y' + x = 0, \text{ when } x > 0.$$

(Ledtråd: Låt $u=y'$ och lös för u först.) (3p) ok!

(5) Find the Taylor polynomial of degree 4 for $\sin^2 x$ about $x = 0$ and use it to calculate

$$\lim_{x \to 0} \frac{3\sin^2 x - 3x^2}{4x^4}$$

(3p)

ok!

(6) (a) Does

$$\sum_{n=0}^{\infty} \frac{n}{n^2+1}$$

converge or diverge? (1p)

ok!

(b) Does

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$$

(Ledtråd: $\sin x \leq x \ \forall x > 0$)

converge or diverge? (2p)

ok!

(7) Let $y(x)$ be the solution to the initial value problem $y' = x - y^2$ such that $y(0) = 0$. Use the Euler method with step size $h = 0.2$ to approximate $y(1)$. (3p)

ok!

(8) (a) (MA059G) Use the definition of derivative to show that $f(x) = \sin x$ is differentiable at $x = 0$. (3p)

ok!

(b) (MA060G) Do 3 iterations, that is calculate x_3, of the Newton-Raphson method to approximate the root of $e^x = 2 - x$ using $x_0 = 0$. (3p)

Good luck!

Test tenta 1

1a) Give the formal definition of "the function f is continuous at $x=a$"

Solution $\lim_{x \to a} f(x) = f(a)$

b) Is the function defined as $f(x)$

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x < 0 \end{cases}$$ continuous

at the point $x = 0$

Solution No, since

$\lim_{x \to 0^-} f(x) = 1 \neq 0 = f(0)$

2) Is the integral $\int_1^\infty xe^{-x}\,dx$ convergent or divergent? If it is convergent calculate the integrals value.

Solution

$$\int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx =$$
$$= -xe^{-x} - e^{-x}$$

$$\lim_{N \to \infty} \int_1^N xe^{-x}\,dx = \lim_{N \to \infty} 1e^{-1} + e^{-1} - Ne^{-N} - e^{-N} = 2e^{-1}$$

The integral is convergent and $\int_1^\infty xe^{-x}\,dx = 2e^{-1}$

3) An orchard has 60 trees and produce an average of 800 apples per ~~trees~~ per year. If more trees are planted the yield per tree will drop. For each tree planted the average yield per tree is reduced by 10 apples per year. How many more trees should be planted in order to maximize the yield from the orchard and what is the maximal yield.

Solution

x = # of trees planted

$Y(x)$ = total yield if x trees are planted

$$Y(x) = (60+x)(800-10x)$$

$$0 \leq x \leq 80$$

$$Y'(x) = 800-10x-10(60+x) = 200-20x$$

$Y'(x) = 0$ when $x = 10$

$$Y(0) = 60 \cdot 800 = 48000$$

$$Y(10) = 70 \cdot 700 = 49000$$

$$Y(80) = 140 \cdot 0 = 0$$

Planting 10 trees will give the maximal yield of 49000 apples per year from the orchard

4) Find all solutions of

$$xy'' + y' + x = 0, \text{ when } x > 0$$

Solution

$$xy'' + y' = -x$$

$$u = y'$$

$$xu' + u = x$$

$$\Leftrightarrow \quad u' + \frac{1}{x}u = 1$$

Int. factor $\quad e^{\int \frac{1}{x}dx} = e^{\ln x (+C)} = x$

$$\frac{d}{dx}(xu) = xu' + u = x$$

$$xu = \frac{x^2}{2} + C$$

$$y' = u = \frac{x}{2} + \frac{C}{x}$$

$$y = \int \frac{x}{2} + \frac{C}{x}\,dx = \frac{x^2}{4} + C\ln x + D$$

5) Find the Taylor polynomial of degree 4 for $\sin^2 x$ about $x=0$ and use it to calculate

$$\lim_{x \to 0} \frac{3\sin^2 x - 3x^2}{4x^4}$$

$f(x) = \sin^2 x$

$f'(x) = 2\sin x \cos x$

$f''(x) = 2(\cos^2 x - \sin^2 x) = 2\cos^2 x - 2\sin^2 x$

$f^{(3)}(x) = -4\cos x \sin x - 4\sin x \cos x = -8\sin x \cos x$

$f^{(4)}(x) = -8(\cos^2 x - \sin^2 x)$

$f(0) = 0 \quad f'(0) = 0 \quad f''(0) = 2$

$f^{(3)}(0) = 0 \quad f^{(4)}(0) = -8$

$$\sin^2 x = \frac{2}{2} x^2 - \frac{8}{4!} x^4 + O(x^5) =$$

$$= x^2 - \frac{1}{3} x^4 + O(x^5)$$

Quicker

$$\sin^2 x = (\sin x)^2 = \left(x - \frac{x^3}{3!} + O(x^5)\right)^2$$

$$= x^2 - \frac{2x^4}{3!} + O(x^5) = x^2 - \frac{x^4}{3} + O(x^5)$$

$$\lim_{x \to 0} \frac{3\sin^2 x - 3x^2}{4x^4} = \lim_{x \to 0} \frac{3x^2 - x^4 + O(x^5) - 3x^2}{4x^4}$$

$$= -\frac{1}{4}$$

6) a) Does $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ converge or diverge?

~~$\lim_{n \to \infty} a_n = \lim_{n \to \infty}$~~ Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges

$a_n = \frac{n}{n^2+1} \qquad b_n = \frac{1}{n}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^2+1} = 1 > 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{n}{n^2+1} \text{ diverges}$$

b) Does $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converge or diverge?

$0 \le \sin\left(\frac{1}{n^2}\right) \le \frac{1}{n^2} \quad n \ge 1$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

So $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges.

7) Let $y(x)$ be the solution to the initial value problem $y' = x + y^2$ such that $y(0) = 0$. Use the Euler Method with step size $h = 0.2$ to approximate $y(1)$.

Solution:

$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0{,}2 \qquad y_1 = y_0 + f(x_0, y_0)h = 0$

$x_2 = 0{,}4 \qquad y_2 = 0 + 0{,}2(0{,}2 - 0^2) = 0{,}04$

$x_3 = 0{,}6 \qquad y_3 = 0{,}04 + 0{,}2(0{,}4 - 0{,}04^2) = 0{,}11968$

$x_4 = 0{,}8 \qquad y_4 = 0{,}11968 + 0{,}2(0{,}6 - 0{,}11968^2) = 0{,}23681\ldots$

$x_5 = 1 \qquad y_5 = 0{,}23681\ldots + 0{,}2(0{,}8 - 0{,}23681\ldots^2) = 0{,}3855990\ldots$

8a (MA0599)

Use the definition of derivative to show that $f(x) = \sin x$ is differentiable at $x = 0$

Solution $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sin x}{x} = 1$

b) Do 3 iterations of the Newton-Raphson Method to approximate the root of $e^x = 2 - x$ using $x_0 = 0$

Solution $f(x) = e^x + x - 2 (= 0)$ $f'(x) = e^x + 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} + x_n - 2}{e^{x_n} + 1}$$

$$x_0 = 0 \qquad x_1 = 0 - \frac{e^0 + 0 - 2}{e^0 + 1} = \frac{1}{2}$$

$$x_2 = \frac{1}{2} - \frac{e^{\frac{1}{2}} + \frac{1}{2} - 2}{e^{\frac{1}{2}} + 1} = 0{,}443851\ldots$$

$$x_3 = x_2 - \frac{e^{x_2} + x_2 - 2}{(e^{x_2} + 1)} = 0{,}442854\ldots$$