Mid Sweden University
NAT
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Testtenta in mathematics.
Fördjupningskurs i analys / Calculus II (MA059G/MA060G).
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Time for tenta: 5 hours
Allowed calculators and collections of formulas: Nonsymbolic calculators and approved "gymnasieformelsamling".

You must provide complete solutions. Motivations and calculations should be comprehensible and easy to follow. One problem per sheet and write only on one side of the paper.

The following guidelines will be used to grade: A 22 p, B 18 p, C 14p, D 10p, E 9p. The grade will be set by how satisfactory you meet the "lärandemål".

ok!

(1) (a) Give the formal definition of the statement: "The function $f(x)$ is differentiable at the point $x = a$". (1p)

ok!

(b) Is the function defined as $f(x) = \begin{cases} x^3 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$ differentable at the point $x = 0$? (2p)

ok!

(2) Is the integral

$$\int_2^\infty \frac{1}{x \ln x}\, dx$$

convergent or divergent? If it is convergent calculate the integrals value. (3p)

ok!

(3) A box is to be made from a rectangular sheet of cardboard 70 cm by 150 cm by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. The box has no top. What is the largest possible volume of the box? (3p)

ok!

(4) Find the solution of

$$y' + 2xy = 2x,$$

such that $y(0) = 0$. (3p)

ok!

(5) Find the Taylor polynomial of degree 4 for e^{x^2} about $x = 0$ and use it to calculate

$$\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2}{2x^4}$$

(3p)

ok!

(6) (a) Does

$$\sum_{n=0}^{\infty} \frac{n}{n+1}$$

converge or diverge? (1p)

ok!

(b) Does

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$$

converge or diverge? (2p)

ok!

(7) Let $y(x)$ be the solution to the initial value problem $y' = x^2 - y^2$ such that $y(0) = 0$. Use the improved Euler method with step size $h = 0.25$ to approximate $y(1)$. (3p)

ok!

(8) (a) (MA059G) Use the Mean-Value Theorem to show that $e^x > x + 1$ for all $x > 0$. (3p)

ok!

(b) (MA060G) Approximate

$$\int_0^1 e^{x^2}\, dx$$

using the Trapeziod Method by dividing the interval $0 \leq x \leq 1$ into 4 subintervals. (3p)

Good luck!

Test Henta 2

1(a) Give formal definitions of the statement "The function $f(x)$ is differentiable at $x=a$"

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists}$$

b) Is $f(x) = \begin{cases} x^3 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$ differentiable at $x=0$?

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^3}{x} = 0$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0-0}{x} = 0 \quad \text{OK!}$$

2) Is the integral $\int_2^\infty \frac{1}{x \ln x} dx$ convergent or divergent? If conv calculate it's value

Solution

$$\int \frac{1}{x \ln x} dx = \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \right] = \int \frac{1}{u} du = \ln u + C = \ln(\ln x) + C$$

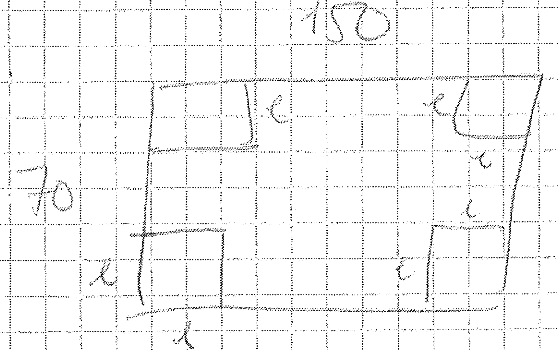
$$\int_2^\infty \frac{1}{x \ln x} dx = \lim_{N\to\infty} \int_2^N \frac{1}{x \ln x} dx =$$

$$= \lim_{N\to\infty} \ln(\ln N) - \ln(\ln 2) = \infty$$

The integral is divergent.

(3) A box is made from a rectangular sheet of cardboard 70 cm by 150 cm by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. The box has no top. What is the largest possible volume of the box?

Solution



$$V(l) = (70-2l)(150-2l)\,l \quad (0 \le l \le 35)$$

$$= 4l^3 - 440l^2 + 70\cdot 150\, l$$

$$V'(l) = 12l^2 - 880l + 70\cdot 150$$

$$l^2 - \frac{880}{12}l + \frac{70\cdot 150}{12} =$$

$$= \left(l - \frac{880}{24}\right)^2 - \left(\frac{880}{24}\right)^2 + \frac{70\cdot 150}{12} = 0$$

$$l = \frac{880}{24} \pm \sqrt{\left(\frac{880}{24}\right)^2 - \frac{70\cdot 150}{12}}$$

$$= \frac{880}{24} \pm \sqrt{\frac{774400 - 504000}{576}}$$

$$= \frac{880}{24} \pm \sqrt{\frac{270400}{576}} = \frac{880}{24} \pm \frac{520}{24}$$

$l_1 = \frac{1400}{24} \approx 58\frac{1}{3}$ $l_2 = \frac{360}{24} = 15$

$V(0) = 0 = V(35)$

$V(15) = (70-30)(150-30) \cdot 15 \text{ cm}^3 =$

$= 72\,000 \text{ cm}^3$

(4) Find the solution of

$y' + 2xy = 2x$

such that $y(0) = 0$.

Int. factor $e^{\int 2x\,dx} = e^{x^2 (+C)}$

$\frac{d}{dx}(e^{x^2} y) = e^{x^2} y' + 2xe^{x^2} y = 2xe^{x^2}$

$e^{x^2} y = \int 2xe^{x^2} dx = \left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array} \right] =$

$= \int e^u du = e^u + D = e^{x^2} + D$

$y = 1 + De^{-x^2}$

$y(0) = 1 + D = 0$ $D = -1$

$y(x) = 1 - e^{-x^2}$

5 Find the Taylor polynomial of degree 4 for e^{x^2} about $x=0$ and use it to calculate

$$\lim_{x\to 0} \frac{e^{x^2}-1-x^2}{2x^4}$$

$f(x) = e^{x^2}$ $\quad f'(x) = 2xe^{x^2}$

$f''(x) = 2e^{x^2} + 4x^2 e^{x^2} = (2+4x^2)e^{x^2}$

$f'''(x) = 8x\, e^{x^2} + (4x+8x^3)e^{x^2} = (12x+8x^3)e^{x^2}$

$f^{(4)}(x) = (12+24x^2)e^{x^2} + (24x^2+16x^4)e^{x^2}$

$f(0)=1 \quad f'(0)=0 \quad f''(0)=2 \quad f'''(0)=0$

$f^{(4)}(0) = 12$

$e^{x^2} = 1 + x^2 + \frac{12}{4!}x^4 + O(x^5) = 1 + x^2 + \frac{1}{2}x^4 + O(x^5)$

$$\lim_{x\to 0} \frac{e^{x^2}-1-x^2}{2x^4} = \lim_{x\to 0} \frac{1+x^2+\frac{x^4}{2}-1-x^2+O(x^5)}{2x^4} = \frac{1}{4}$$

6 a) Does $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverge or converge?

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ so

$\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges

b) Does $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ diverge or converge?

We do the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^3}{(3n+3)!} \cdot \frac{3n!}{n!} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty} \frac{n^3(1+\frac{1}{n})^3}{n^3(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})}$$

$= \frac{1}{3^3} < 1$ so ~~the~~

$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ is convergent.

7) Let $y(x)$ be the solution of the initial value problem $y' = x^2 - y^2$, $y(0) = 0$. Use the improved Euler method with step size $h = 0{,}25$ to approximate $y(1)$.

$x_0 = 0 \qquad y_0 = 0$

$x_1 = 0{,}25 \qquad u_1 = y_0 + h f(x_0, y_0) = 0$

$y_1 = y_0 + h\left(\frac{f(x_0,y_0) + f(x_1,u_1)}{2}\right) = 0 + 0{,}25\left(\frac{0 + 0{,}25^2}{2}\right)$

$= 0{,}0078125$

$x_2 = 0{,}5 \qquad u_2 = y_1 + h(x_1^2 - y_1^2) = y_1 + 0{,}25(0{,}25^2 - y_1^2) =$

$= 0{,}0234222$

$y_2 = y_1 + h\left(\frac{x_1^2 - y_1^2 + x_2^2 - u_2^2}{2}\right) = 0{,}046798$

$x_3 = 0{,}75 \qquad u_3 = y_2 + h(x_2^2 - y_2^2) =$

$= 0{,}108751$

$y_3 = y_2 + h\left(\frac{x_2^2 - y_2^2 + x_3^2 - u_3^2}{2}\right) = 0{,}146009$

$x_4 = 1$ $\quad u_4 = y_3 + h(x_3^2 - y_3^2) = 0,281866\ldots$

$$y_4 = y_3 + h\left(\frac{x_3^2 - y_3^2 + x_4^2 - u_4^2}{2}\right) =$$

$$= 0,329304\ldots$$

8a (MA059)

Use the Mean Value Theorem to show that $e^x > x+1$ for $x>0$

~~If~~ ~~not~~ (First $e^1 > 2$)

~~If exists~~ If not exist a such that $e^a = a+1$

Standing $f(x) = e^x - x - 1$ differentiable

$f(0) = f(a) = 0$

Exist $0 < s < a$ so $f'(s) = 0$

$$e^s = 1 \Rightarrow s = 0$$

8b (MA060G)

Approximate $\int_0^1 e^{x^2}\,dx$ using the Trapezoid Method by dividing the interval $0 \leq x \leq 1$ into 4 subintervals

$$\int_0^1 e^{x^2}\,dx \approx 0,25\left(\frac{e^0}{2} + e^{0,25} + e^{0,5} + e^{0,75} + \frac{e^1}{2}\right) =$$

$$= 1,727219\ldots$$