MA014G
Algebra and Discrete Mathematics A
Suggested Solutions to Assignment 5

Question 1 Let n be an integer. The contrapositive of the proposition

'**If** n^2 is odd **then** n is odd'

is the proposition

'**If** n is even **then** n^2 is even.'

The proposition is proved by proving its contrapositive by a direct proof:

Suppose that n is even, then there exists an integer s such that $n = 2s$. Thus

$$n^2 = (2s)^2 = 2(2s^2),$$

which shows that n^2 is even as it is divisible by 2.

Question 2.
We put 201 marbles into 50 boxes, and we want to show (by contradiction) that there is at least one box containing five or more marbles.

So assume the opposite, i.e. assume that no box contains five or more marbles. This means that every box has either 0, 1, 2, 3 or 4 marbles in it. But if there are at most 4 marbles in each of the 50 boxes, then the total number of marbles in the boxes is at most $4 \cdot 50 = 200$. This contradicts the fact that we put 201 marbles into the boxes.
Hence we may conclude that our assumption is false, and so its opposite will be true, i.e there will be at least one box having five or more marbles in it.

Question 3.
Substitute $a_i = t^i$ for all i to get $t^{n+2} = -6t^{n+1} - 9t^n$. Thus the characteristic equation is

$$t^2 + 6t + 9 = 0.$$

This has roots $\alpha_1 = -3$ and $\alpha_2 = -3$, so the general solution is

$$a_n = A(-3)^n + Bn(-3)^n \text{ where } A \text{ and } B \text{ are constants.}$$

From the two initial conditions we get:
$-1 = a_0 = A(-3)^0 = A$ and $1 = a_1 = A(-3)^1 + B(-3)^1 = (-1)(-3) + (-3)B$.
Solving these two equations in two unknowns yields $A = -1$ and $B = \frac{2}{3}$ and so the solution is

$$a_n = -(-3)^n + \tfrac{2}{3}n(-3)^n = (\tfrac{2}{3}n - 1)(-3)^n = (3 - 2n)(-3)^{n-1} \quad \text{for } n \geq 0.$$

Question 4.
We consider the sequence $\{s_n\}_{n=1}^{\infty}$ defined by the recurrence relation $s_n = s_{n-1} + n^2$ for $n \geq 2$, and the initial term $s_1 = 1$.

(a)
$s_2 = s_{2-1} + 2^2 = s_1 + 4 = 1 + 4 = 5$;
$s_3 = s_{3-1} + 3^2 = s_2 + 9 = 5 + 9 = 14$;
$s_4 = s_{4-1} + 4^2 = s_3 + 16 = 14 + 16 = 30$;
$s_5 = s_{5-1} + 5^2 = s_4 + 25 = 30 + 25 = 55$.

(b) We prove by induction that $s_n = \sum_{r=1}^{n} r^2$ for all $n \geq 1$:

Base case:

For $n = 1$, the $LHS = s_1 = 1$, and the $RHS = \sum_{r=1}^{1} r^2 = 1^2 = 1$. So the result is true for $n = 1$.

Inductive hypothesis: Assume that $s_k = \sum_{r=1}^{k} r^2$ for some $k \geq 1$.

Inductive step:

We must prove $s_{k+1} = \sum_{r=1}^{k+1} r^2$.

But

$$\begin{aligned} s_{k+1} &= s_k + (k+1)^2 \text{ by the recurrence relation} \\ &= \sum_{r=1}^{k} r^2 + (k+1)^2 \text{ by the inductive hypothesis} \\ &= \sum_{r=1}^{k+1} r^2, \end{aligned}$$

whence the result is true for $n = k + 1$ also and thus for all $n \geq 1$ by induction.

(c) We prove by induction that $s_n = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$:

Base case:

For $n = 1$, the $LHS = s_1 = 1$, and the $RHS = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$. So the result is true for $n = 1$.

Inductive hypothesis: Assume that $s_k = \frac{k(k+1)(2k+1)}{6}$ for some $k \geq 1$.

Inductive step:

We must prove $s_{k+1} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$.

But

$$\begin{aligned} LHS = s_{k+1} &= s_k + (k+1)^2 \text{ by the recurrence relation} \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ by the inductive hypothesis} \\ &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \end{aligned}$$

Now

$$\begin{aligned} RHS &= \frac{(k+1)(k+2)(2(k+1)+1)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= LHS, \end{aligned}$$

whence the result is true for $n = k + 1$ also and thus for all $n \geq 1$ by induction.