MA014G
Algebra and Discrete Mathematics A

Lecture Notes 1

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Pia Heidtmann
070811

Sets (mängder)

A **set** is a collection of objects such that we can tell whether a given object is in the set or not.

The objects are called the elements or members of the set.

Example

- The set of integers (heltal) between -3 and 3 is

$$S = \{-3, -2, -1, 0, 1, 2, 3\}$$

- Each number in the set is an element of the set. For example, 2 is an element of S. In symbols we write

$$2 \in S.$$

A set has <u>no order</u> and repeated elements are disregarded, so for example

The set $\{1, 3, 5\}$ is equal to the set $\{3, 1, 5\}$.
The set $\{1, 3, 3, 5\}$ is equal to the set $\{1, 3, 5\}$.

Two sets are equal if they have exactly the same elements.

The number of distinct elements of a set is called its cardinality. If a set A contains exactly n different elements, we write

$$|A| = n$$

Example

- The sets $\{a, b, p\}$ and $\{p, b, a\}$ are equal since they have exactly the same elements. Both have cardinality 3.

- The sets $A = \{a, b, b, p\}$ and $B = \{a, b, p\}$ are equal. We write $A = B$.
  In $\{a, b, b, p\}$ the element b appears twice, but we only count it once, so

$$|A| = |B| = 3.$$

- The sets $\{5\}$ and $\{\{5\}\}$ are NOT equal. The first set consists of the number 5 while the second consists of the set $\{5\}$ and these two things are not the same.

  However, both sets have cardinality 1:

$$|\{5\}| = |\{\{5\}\}|$$

BUT $$\{5\} \neq \{\{5\}\}$$

Venn diagrams for sets

It is often useful to be able to draw a picture of a set. The nineteenth century English mathematician John Venn invented a method for doing so:

When we are working with sets, they usually all have got elements from some underlying universal set. If for example all elements from a set are integers (heltal), the universal set would be the set of all integers, $\mathbb{Z}$.

In Swedish the universal set is called a grundmängd, and we shall therefore call our universal set G.

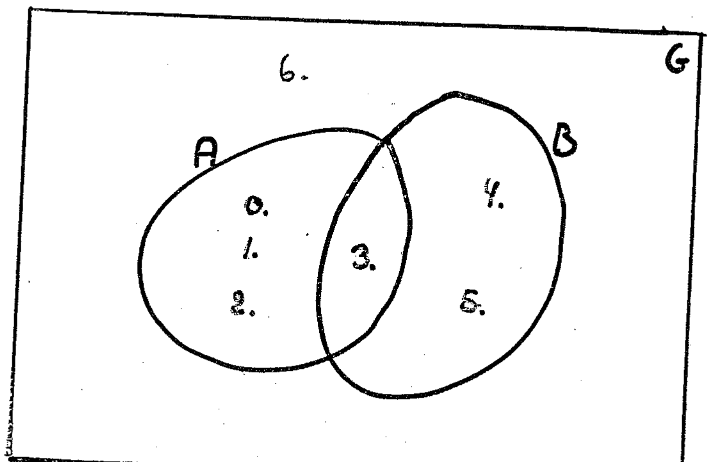
In a Venn diagram, we draw the universal set as a big rectangle and the sets we work with are drawn as closed regions (usually circles or ovals) *inside* the rectangle.

EXAMPLE

Let $G = \{0,1,2,3,4,5,6\}$ and consider the two sets

$$A = \{0,1,2,3\} \qquad B = \{3,4,5\}$$

The Venn diagram for this scenario is here:



Subsets (delmängder)

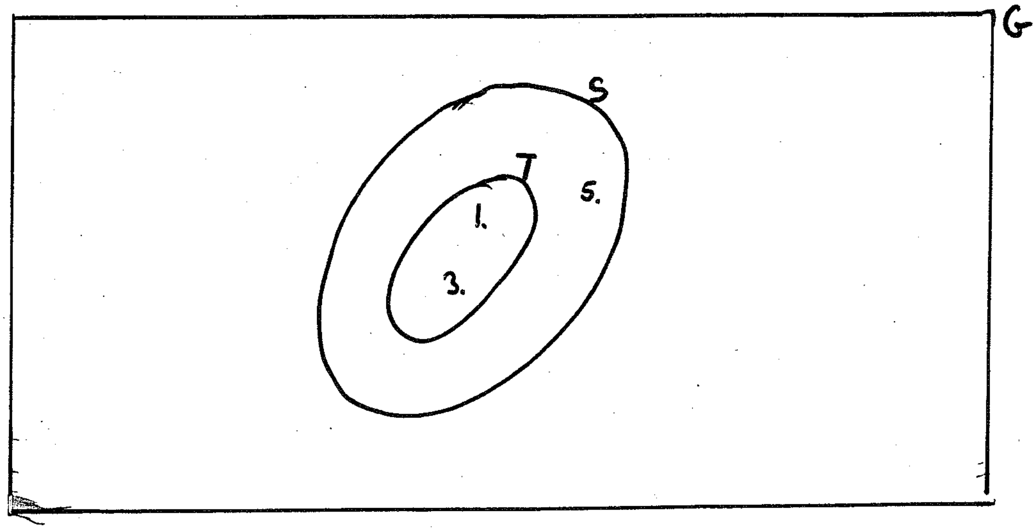
A subset T of a set S, written $T \subseteq S$, is a set, all of whose elements are from the set S.

EXAMPLE

The set $\{1,3\}$ is a subset of the set $\{1,3,5\}$.

$$\{1,3\} \subseteq \{1,3,5\}$$

The Venn diagram for this is



- Any set is a subset of itself, e.g. $\{1,3,5\} \subseteq \{1,3,5\}$.
- Any set has the empty set, $\{\ \}$, as a subset, e.g. $\{\ \} \subseteq \{1,3,5\}$. The empty set is usually denoted $\emptyset$, so

$$\{\ \} = \emptyset \quad \text{and} \quad |\emptyset| = 0$$

- A subset of a set S which is NOT S *itself* is called a proper subset (äkta delmängd) of S
- In the example above $\{1,3\}$ is a proper subset of $\{1,3,5\}$, we write

$$\{1,3\} \subset \{1,3,5\}$$

OPERATIONS ON SETS

Union

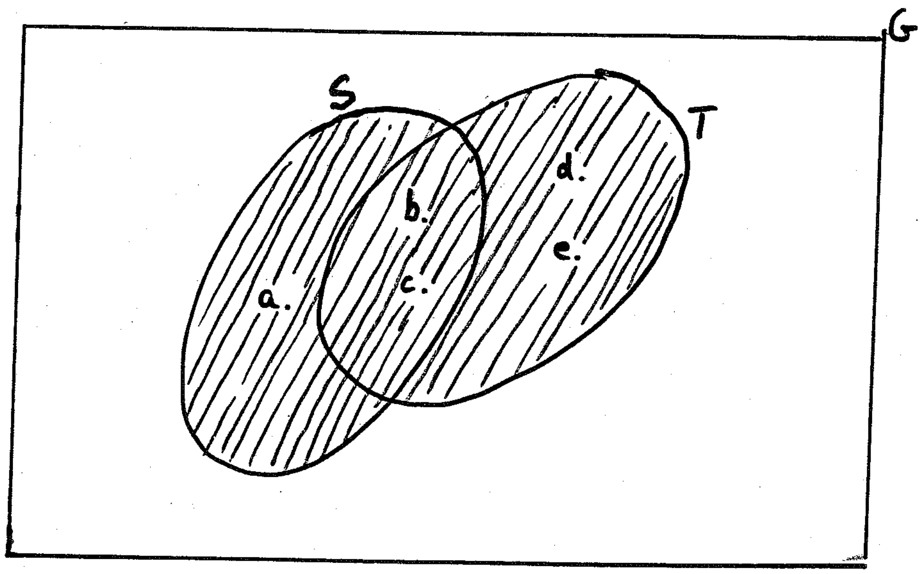
The union of sets S and T, written $S \cup T$, is the set of all elements which are in either of the sets S or T or in both.

EXAMPLE

If $S = \{a, b, c\}$ and $T = \{b, c, d, e\}$ then

$$S \cup T = \{a, b, c, d, e\}$$

The Venn diagram is



Intersection (snittet)

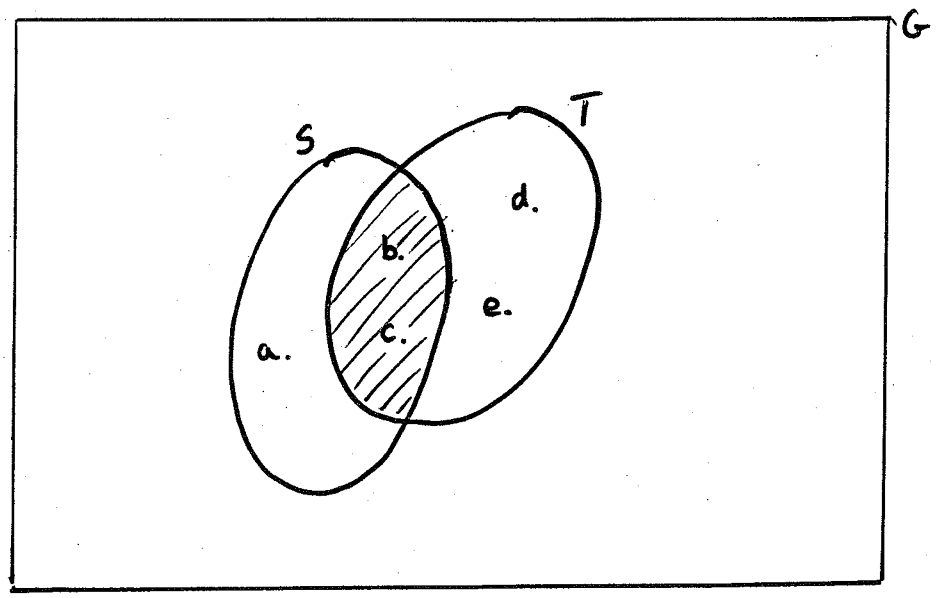
The intersection of sets S and T, written $S \cap T$, is the set of elements which are in both the sets S and T.

EXAMPLE

If $S = \{a, b, c\}$ and $T = \{b, c, d, e\}$ then

$$S \cap T = \{b, c\}$$

The Venn diagram is



Set difference

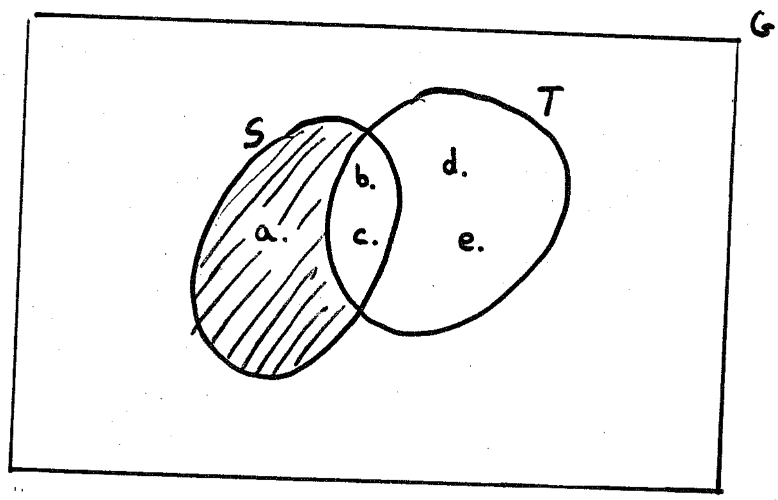
The difference of the sets S and T, written $S-T$, is the set of elements which are in the set S but not in T.

EXAMPLE

If $S = \{a, b, c\}$ and $T = \{b, c, d, e\}$ then

$$S - T = \{a\}$$

The Venn diagram is:



Note that generally, as for numbers,

$$S-T \neq T-S$$

e.g. in the above example

$$S-T = \{a\}$$

$$T-S = \{d, e\}$$

Set complement

The complement $\overline{S}$ of a set S consists of all elements in the universal set G which are **not** in S.

That is : $\overline{S} = G - S$

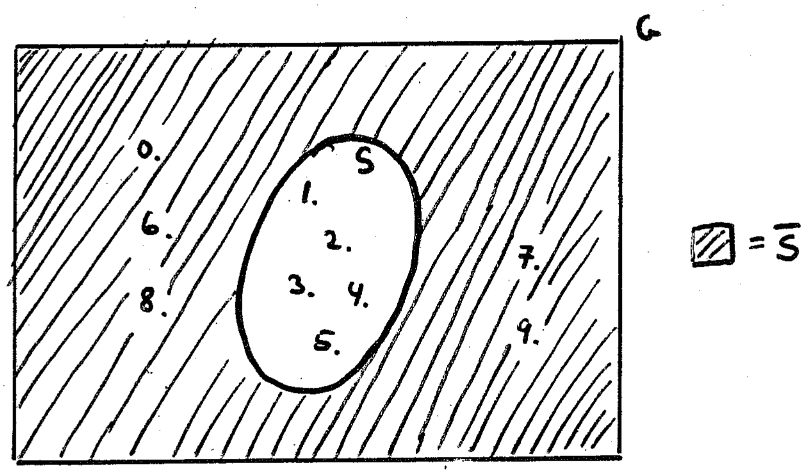
EXAMPLE

Suppose that $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

and that $S = \{1, 2, 3, 4, 5\}$

Then $\overline{S} = \{0, 6, 7, 8, 9\}$

The Venn diagram looks like this:



Some universal sets of numbers

The set of positive integers is

$$\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$$

NOTE that 0 is NOT a positive integer.

The set consisting of positive integers <u>and</u> 0 is the set of natural numbers (naturliga tal)

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

The set of negative integers is

$$\mathbb{Z}_- = \{-1, -2, -3, \ldots\}$$

NOTE that 0 is NOT a negative integer.

The set of all INTEGERS is called $\mathbb{Z}$.

$$\mathbb{Z} = \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+ .$$

Describing sets

For many sets, the easiest way of describing them is just to list their elements as we have done in the examples above. For example, the set of even integers between 1 and 15 are:

$$\{2, 4, 6, 8, 10, 12, 14\}.$$

But sometimes listing all elements of a set is too much work. For example, the set of even integers between 1 and 1500 has 750 elements, and we cannot list them all. We write this set as

$$\{2, 4, 6, 8, 10, \ldots, 1500\},$$

where the symbol "..." is read "and so on". We can use the '...'- symbol whenever it is clear which elements we mean.

Another way of describing the same set would be:

$$\{2n \mid n \in \mathbb{Z},\ 1 \leq n \leq 750\}$$

or $$\{x \in \mathbb{Z} \mid x = 2n,\ n = 1, 2, \ldots, 750\}$$

or $$\{x \in \mathbb{Z} \mid 1 \leq x \leq 1500,\ x \text{ is even}\}$$

Combining set operations

Just as ordinary arithmetical operations on numbers, like e.g. +, satisfy various laws, like for example the associative law for addition which states that

$$(a+b)+c = a+(b+c)$$

for all integers a, b and c; we also have similar rules for combining set operations.

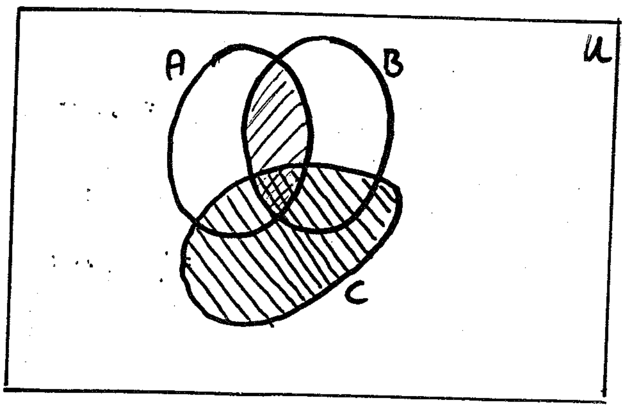
We shall go over the most important ones here, but you can find others in the book (Thm. 2.1.10 p.59 in [3]).

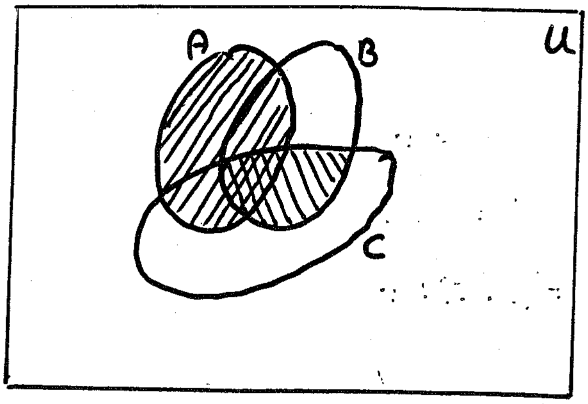
The associative laws for union and intersection.

Let A, B and C be subsets of a universal set U, then

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Such identities can easily be illustrated by drawing Venn diagrams, e.g.





$A \cap B =$ ////

$C =$ \\\\\\\\

$(A\cap B)\cap C =$

$A =$ ////

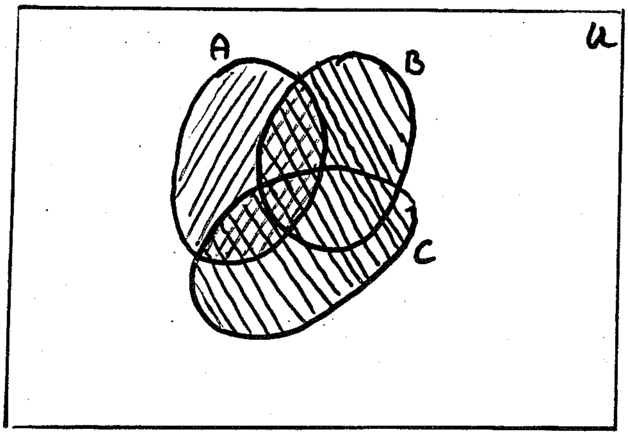
$B \cap C =$ \\\\\\\\

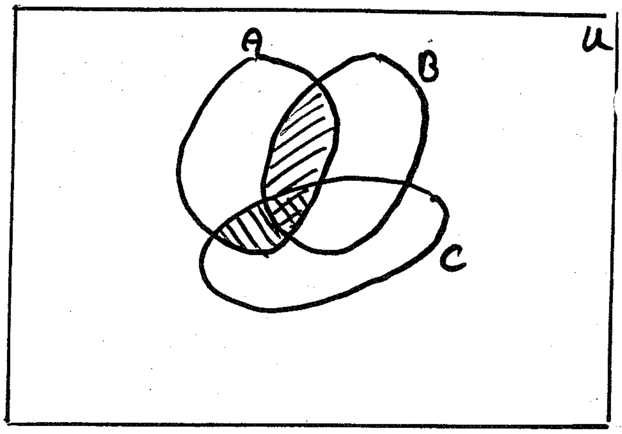
$A \cap (B \cap C) =$

The Distributive Laws

Let A, B and C be subsets of the universal set $\mathcal{U}$, then

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



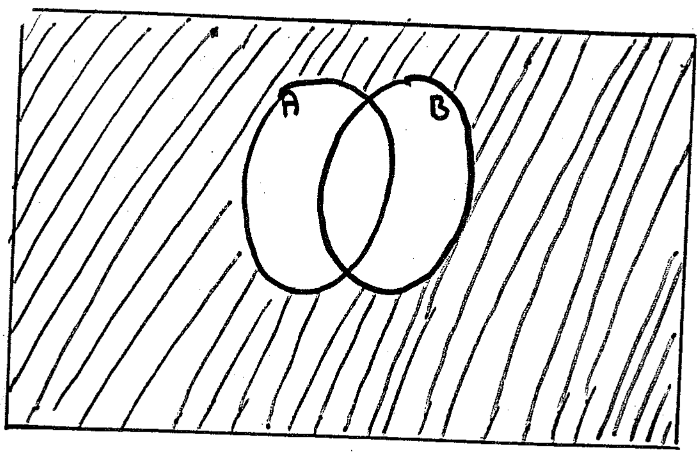


$A \cap (B \cup C) \quad = \quad (A \cap B) \cup (A \cap C).$

De Morgan's Laws

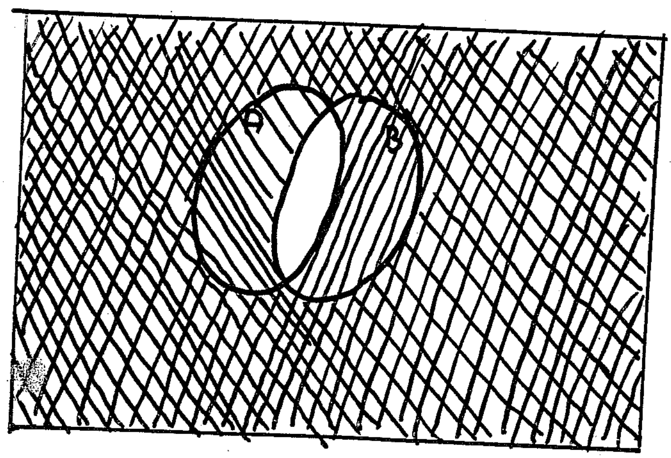
Let A and B be subsets of the universal set U, then

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$



$\overline{A \cup B}$

$=$



$\overline{A} \cap \overline{B}$

Counting Things

If we toss a coin five times how many sequences of heads and tails can arise?

In order to find this we use the *multiplication principle.*

In this case we have five events recorded and for each there are two possible outcomes.

Each different sequence of results is counted separately; for example (heads, heads, tails, tails, heads) and (heads, tails, heads, heads, tails) are different outcomes.

The total number can thus be determined by looking at the number of options for each and multiplying these together. In this case it is

$$2 \times 2 \times 2 \times 2 \times 2 = 32$$

The **multiplication principle** states that if we have a procedure with n stages and each stage r has a_r possible outcomes then the total number of possible outcomes is

$$\underline{a_1 \times a_2 \times a_3 \times \ldots \times a_n}$$

Example

On my shelf I have 12 different mathematics books and 6 different computing books.

In how many ways can I select two books, one from each subject, to take on a weekend trip?

Example

In my local pizzeria they have 52 different pizzas on the menu, 8 kinds of drinks in their fridge and two kinds of sallad.
I how many ways can I choose a meal consisting of one pizza, one drink and one sallad?

Example

Suppose we roll two dice. One die is blue and one die is red.

How many possible rolls are there?

How many outcomes are possible where the red die shows 6?

How many outcomes are possible where the red die shows an even number and the blue die shows an odd number?

Example

There are 2^n n-bit binary strings.

Proof.
We can find any n-bit binary string in n stages:

Stage 1: Choose the 1'st bit in the string
Stage 2: Choose the 2'nd bit in the string
⋮
Stage i: Choose the i'th bit in the string
⋮
Stage n: Choose the n'th bit in the string.

At each stage we have precisely 2 choices, either the bit is 0 or the bit is 1, and each choice is independent of the previous choices.

The multiplication principle with $a_1 = a_2 = a_3 = \ldots = a_n = 2$ thus gives us that there are 2^n n-bit binary strings. ■

Some More Sets

Let A and B be subsets of the universal set $\mathcal{U}$.
The Cartesian product of A and B, written $A \times B$, consists of all *ordered* pairs (a, b) where $a \in A$ and $b \in B$.

Example
Let $A = \{1, 2\}$ and $B = \{a, b\}$, then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

Note that in general $A \times B \neq B \times A$. In our example

$$B \times A = \{(a, 1), (b, 1), (a, 2), (b, 2)\}.$$

Using the multiplication principle we find the cardinality

$$|A \times B| = |A||B|.$$

More generally:
Let $A_1, A_2, A_3, \ldots, A_n$ be subsets of the universal set $\mathcal{U}$.
The product set

$$A_1 \times A_2 \times A_3 \times \ldots \times A_n$$

consists of all n-tuples $(a_1, a_2, a_3, \ldots, a_n)$, where $a_i \in A_i$ for $1 \leq i \leq n$.

Using the multiplication principle we find the cardinality

$$|A_1 \times A_2 \times A_3 \times \ldots \times A_n| = |A_1||A_2||A_3| \ldots |A_n|.$$

How do we show that two sets are the same size?

We have two ways:

1. Count the number of elements in each set and check that they are equal.

2. Pair up the elements of the two sets.

   (This is also known as setting up a *1-1 correspondence* between the two sets, we shall learn more about 1-1 correpondences later on in the course.)

Example

The set of all subsets of a set X is called the powerset of X and is denoted $\mathcal{P}(X)$.

If $|X| = n$ then $|\mathcal{P}(X)| = 2^n$.

We can prove this by pairing off the elements of $\mathcal{P}(X)$ with the elements in the set of all n-bit binary strings, and since we know that there are 2^n n-bit binary strings, there will thus also be 2^n elements of $\mathcal{P}(X)$.

Suppose that $X = \{x_1, x_2, \ldots x_n\}$, and let S be any subset of X, then we pair off S with the following unique n-bit binary string:

Bit 1 is 1 if $x_1 \in S$ and 0 if $x_1 \notin S$;
Bit 2 is 1 if $x_2 \in S$ and 0 if $x_2 \notin S$;
$\vdots$
Bit i is 1 if $x_i \in S$ and 0 if $x_i \notin S$;
$\vdots$
Bit n is 1 if $x_n \in S$ and 0 if $x_n \notin S$.

Let us do this for a small example.
Suppose that $X = \{a, b, c\}$, and let S be any subset of X, then we use a 3-bit binary string to code S as follows:

Bit 1 is 1 if $a \in S$ and 0 if $a \notin S$;
Bit 2 is 1 if $b \in S$ and 0 if $b \notin S$;
Bit 3 is 1 if $c \in S$ and 0 if $c \notin S$.

The 1-1 correspondence between subsets and codes is thus

subset	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
code	000	100	010	001	110	101	011	111

Partitions

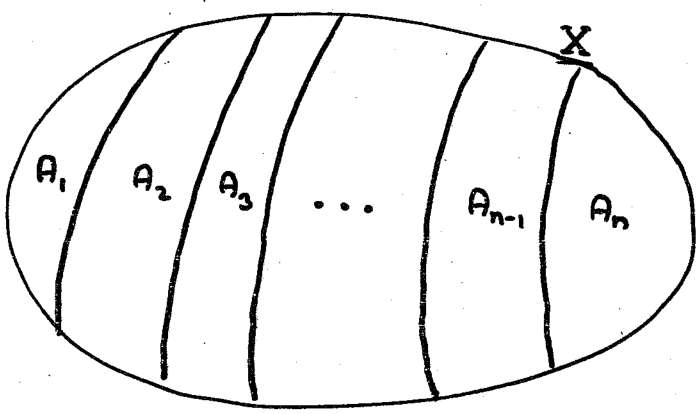
A collection of *non-empty* sets $A_1, A_2, A_3, \ldots, A_n$ is a partition of the set X if the following two conditions hold.

1. The union of all the sets is X, that is $X = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$.
2. The sets are pairwise disjoint, that is $A_i \cap A_j = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq n$.

The n sets $A_1, A_2, A_3, \ldots, A_n$ are known as the parts of the partition, and since each element of X is in exactly one of the A_i, we can conclude that

$$|X| = |A_1| + |A_2| + |A_3| + \ldots + |A_n|.$$

This result is known as the **addition principle.**



Example

Suppose that in a bag of sweets there are 5 Dumle, and 7 mints and 8 jelly beans. How many sweets are there in the bag?

Example

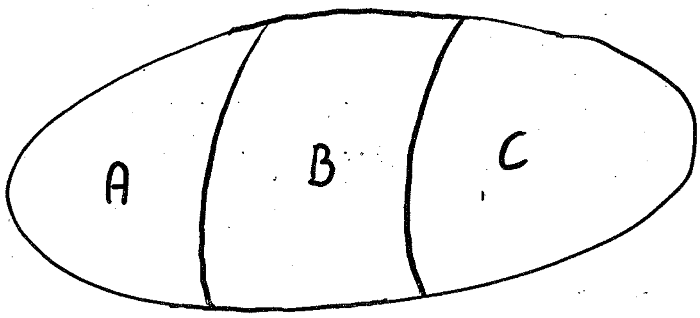
On my shelf I have 5 Maths books, 3 Computing books and 4 Physics books.

In how many ways can I choose a pair of books from different subjects among the books on my shelf?

The solution to this is easily seen, when you note that I have exactly three possibilities for a pair of subjects, namely

A. Maths & Computing;
B. Maths & Physics;
C. Computing and Physics.

Note also that the three categories A, B and C are disjoint.



Using the *multiplication principle*, the selection in A can be done in 5×3 ways.
Similarly there are 5×4 selections in B and 3×4 selections in C.

Hence by the *addition principle* there are thus

$$5 \times 3 \; + \; 5 \times 4 \; + \; 3 \times 4 \; = \; 47$$

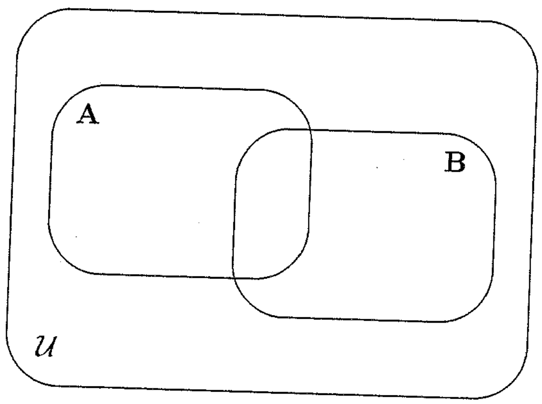
ways of choosing a pair of books from different subjects among the books on my shelf.

The Principle of Inclusion-Exclusion

Suppose we have two sets A and B and we know the sizes of these sets. Do we know the size of the union of these sets?

Unfortunately, we cannot simply add the sizes of the individual sets, as if there are elements that lie in both sets they will be counted twice but will only count as one in the union. Thus for each element that is in both sets, and therefore is in the intersection of the sets, the sum of the sizes of the sets is one too great. Therefore in order to find the size of the union of the sets we have to sum the sizes of the individual sets and then subtract the size of the intersection of the sets. This gives:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Example

If $A = \{a, b, c, d\}$ and $B = \{b, e, f\}$ then:

$$\begin{aligned} A \cup B &= \{a, b, c, d, e, f\} \\ A \cap B &= \{b\} \end{aligned}$$

and

$$|A| = 4, \ |B| = 3, \ |A \cup B| = 6, \ |A \cap B| = 1$$

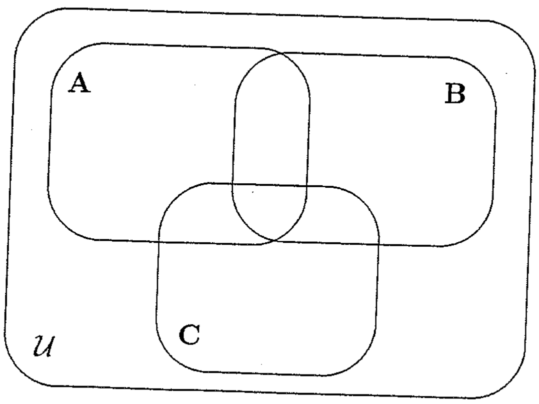
We can therefore see that in this case

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This result extends to three sets:

Proposition B1.3 Let A, B and C be three sets from the same universal set $\mathcal{U}$, not necessarily disjoint. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Example

If in a town 50% of people use buses, 40% of people use trains, and 20% of people use both, how many people use at least one of these forms of transport?

Let $U =$ the set of people in the town.

Let $A =$ the set of people who use buses. $|A| = \frac{50}{100}|U|$.

Let $B =$ the set of people who use trains. $|B| = \frac{40}{100}|U|$,

$|A \cap B| = \frac{20}{100}|U|$.

Then:

$$|A \cup B| = |A| + |B| - |A \cap B| = \frac{50}{100}|U| + \frac{40}{100}|U| - \frac{20}{100}|U| = \frac{70}{100}|U|$$

Thus 70% of people use at least one of these forms of transport.

Example

How many integers in the set

$$\{1, 2, 3, 4, 5, \ldots, 100\}$$

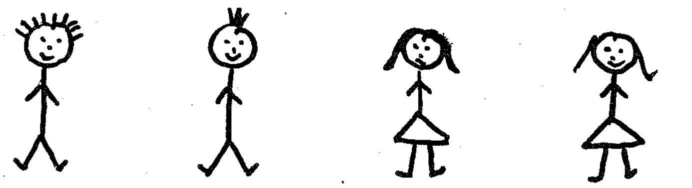
are not divisible by 2, 3 or 5?

ORDERED OR UNORDERED SELECTIONS ?

Example

① I know 4 children. In how many ways can I choose a pair of children where one of them is to help me do the gardening and the other is to help me do the dishes ?

② I know 4 children. In how many ways can I choose a pair of them to take to the cinema ?



Allan Bill Christine Debbie

ORDERED SELECTIONS WITHOUT REPETITION

Problem: Choose an ordered selection of k objects from a set of n objects where repetition is not allowed.

Such selections are called permutations of length k from n objects, and we denote their number by

$$P(n,k).$$

How many are there?

Solution: This is a k-stage problem solvable by the multiplication principle:

STAGE 1 : choose object 1	n	choices
STAGE 2 : choose object 2	$n-1$	choices
STAGE 3 : choose object 3	$n-2$	choices
$\vdots$		
STAGE k : choose object k	$n-(k-1)$	choices

$$\boxed{\text{So} \quad P(n,k) = n\cdot(n-1)\cdot(n-2)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}}$$

The factorial function $n!$ is defined for all natural numbers $n \in \mathbb{N}$:

$0! = 1$, $1! = 1$, $2! = 2\cdot 1$, $3! = 3\cdot 2\cdot 1$, $4! = 4\cdot 3\cdot 2\cdot 1$,

$n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$

UNORDERED SELECTIONS WITHOUT REPETITION

Problem: In how many ways can a subset of k elements be chosen from a set of n elements?

Such selections are called combinations of k elements chosen from n, and we denote their number by

$$\binom{n}{k}$$

read: "n choose k"

You can also sometimes see an alternative notation for this:

$$\binom{n}{k} = C(n,k).$$

How many are there?

ANSWER: $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example

The number of ways of choosing a subset of size 2 from $\{A,B,C,D\}$ is

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\cdot3\cdot2\cdot1}{2\cdot1\cdot2\cdot1} = \frac{4\cdot3}{2} = 6$$

They are:

$\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{B,D\}$, $\{C,D\}$

Proposition There are $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ ways of choosing a subset of k elements from a set of k elements.

PROOF:

Reconsider the problem of choosing an ordered list of k elements without repetition from n elements. How many such lists are there?

① We found previously that the answer is $P(n,k) = \underline{\underline{\frac{n!}{(n-k)!}}}$

② However, we could solve the problem in a different way by following a 2-stage procedure

Ⓐ Choose an unordered set of k elements from the n elements. How many ways are there of doing this? This is the number $\binom{n}{k}$ we are seeking!

Ⓑ Order the set chosen in Ⓐ so that it becomes a list. This can be done in

$$P(k,k) = k\cdot(k-1)\cdots 2\cdot 1 = k! \text{ ways}$$

So by the multiplication principle the number of ordered lists of k elements chosen without repetion from a set of n elements is

$$\underline{\underline{\binom{n}{k}\cdot k!}}$$

Comparing the two solutions from ① and ② we thus have

$$\frac{n!}{(n-k)!} = \binom{n}{k}\cdot k!$$

which yields the required formula for $\binom{n}{k}$. ▨

Examples

① In how many ways can a committee with 12 members choose a chairperson, a treasurer and a secretary?

order? ✓

Answer: $12 \times 11 \times 10$

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② How many rearrangements are there of the word DISKRET?

order? ✓

Answer: $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$

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③ How many 5-digit telephone numbers have a repeated digit?

Answer: There are $9 \cdot 10^4$ 5-digit phone numbers as phone numbers may not start with a zero.

Phone numbers have either got no repeated digits or they have ≥ 1 repeated digits.

$9 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ have no repeated digits, so

$$9 \cdot 10^4 - 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

have at least one repeated digit.

Examples (cont.)

(4) A hand in a game of poker consists of a selection of 5 cards from a deck of 52 cards.

(a) How many poker hands are there?

order? X

Answer : $\binom{52}{5} = \underline{\underline{\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot}}}$

(b) How many where all five cards are of the same suit?

Answer : There are $\underline{\binom{13}{5}}$ hands all hearts

There are $\underline{\binom{13}{5}}$ hands all diamonds

There are $\underline{\binom{13}{5}}$ hands all clubs

There are $\underline{\binom{13}{5}}$ hands all spades.

By the addition principle there are thus

$$4 \cdot \binom{13}{5} = \underline{\underline{\frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2}}}$$

(c) How many with 3 cards from one suit and two cards from another?

STAGE 1 : choose a suit and 3 cards from it : $\underline{\binom{13}{3} \cdot 4}$ selections

STAGE 2 : choose another suit and 2 cards from it : $\underline{\binom{13}{2} \cdot 3}$ selections.

ANSWER : $\underline{\underline{4 \cdot 3 \cdot \binom{13}{3} \cdot \binom{13}{2}}}$