MA014G

Algebra and Discrete Mathematics A

Lecture Notes 2

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Sequences (följder)

A sequence is an ordered list of numbers.

EXAMPLE

A sequence s is given by for example

$$1, 4, 7, 10, 13, 16, 19, 22, 25, \dots$$

This sequence can be described as $s_1 = 1$, $s_2 = 4$, $s_3 = 7$, $s_4 = 10, \dots$

The first term of a sequence is known as the initial term.
And we say that the general term, that is the n'th term, is s_n.

In the example above, the initial term is $s_1 = 1$

and the general term is $s_n = 3n - 2$.

There are three ways in which [J] describes the sequence s in the example above:

- The sequence s is defined by $s_n = 3n-2$, $n = 1, 2, 3, \dots$
- The sequence $\{s_n\}$ defined by the rule $s_n = 3n-2$, $n \geq 1$
- The sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_n = 3n-2$.

The symbol '∞' is read 'infinity' and is not a number, but just a symbol meaning that n can become as large as we like, the sequence is never-ending.

Some examples of sequences

Let a be the sequence given by

$$a_n = n^2+1,$$

for $n \geq 1$. Then a is the infinite sequence

$$2,\ 5,\ 10,\ 17,\ 26,\ 37,\ 50,\ 65,\ 82,\ \dots$$

Let b be the sequence given by

$$b_n = n^2+1,$$

for $1 \leq n \leq 5$. Then b is the finite sequence

$$2,\ 5,\ 10,\ 17,\ 26.$$

$a = \{n^2+1\}_{n=1}^{\infty}$

$b = \{n^2+1\}_{n=1}^{5}$

Some examples of sequences

Let s be the sequence given by

$$s_n = (-1)^n,$$

for $n \geq 1$. Then s is the infinite sequence

$$-1, 1, -1, 1, -1, 1, -1, 1, \ldots$$

Let t be the sequence given by

$$t_n = n!,$$

for $n \geq 1$. Then t is the infinite sequence

$$1, 2, 6, 24, 120, 720, \ldots$$

because

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Note $0! := 1$, $1! := 1$.

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$\vdots$

'$n!$' is read as 'n factorial', in Swedish: 'n fakultet' a sequence s is increasing (växande) if

$$s_n \leq s_{n+1} \quad \text{for all } n.$$

a sequence s is decreasing (avtagende) if

$$s_n \geq s_{n+1} \quad \text{for all } n.$$

Example

The sequence $\{s_n\}_{n=1}^{\infty}$ given by $s_n = 2n-3$ is increasing:

$$-1, 1, 3, 5, 7, 9, 11, \ldots$$

because $\quad s_{n+1} = 2(n+1)-3 = 2n+2-3 = 2n-1$

and $\quad s_n = 2n-3$

so $\quad s_n \leq s_{n+1} \quad$ for all $n \geq 1$.

The sequence $\{t_n\}_{n=1}^{\infty}$ given by $t_n = \left(\frac{1}{2}\right)^n$ is decreasing:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

because $t_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ and $t_n = \left(\frac{1}{2}\right)^n$ so

$$t_n \geq t_{n+1} \quad \text{for all } n \geq 1$$

Note, there are sequences which are neither increasing, nor decreasing:

e.g. $\quad -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, -\frac{1}{64}, \frac{1}{128}, \ldots$

Creating new sequences from old

Suppose that we have a sequence s:

$$s_1, s_2, s_3, s_4, s_5, \dots$$

An easy way of creating a new sequence is to just throw away some of the terms in the sequence s, but keep the order of the terms. Such a new sequence is called a subsequence (delföljd) of s.

EXAMPLE

Suppose the sequence s is given by

$$s_n = n^2, \quad \text{for } n \geq 1:$$

$$1, 4, 9, 16, 25, 36, 49, 64, \dots$$

If we throw away all the odd terms in this sequence, we get the subsequence

$$4, 16, 36, 64, 100, 144, \dots$$

The subsequence is the sequence $\{t_n\}_{n=1}^{\infty}$ where $t_n = (2n)^2$.

More formally:

Let $\{s_n\}$ be a sequence defined for $n = m, m+1, m+2, \dots$

Let $n_1, n_2, n_3, \dots$ be an increasing sequence satisfying that all $n_1, n_2, n_3, \dots \in \{m, m+1, m+2, \dots\}$.

Then $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$, $k = 1, 2, 3, \dots$

Another way of creating new sequences from old is by adding the terms of two known sequences:

EXAMPLE

Let u be the sequence

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \ldots$$

that is $u_n = 2n$ for $n = 1, 2, 3, \ldots$

Let v be the sequence

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots$$

that is $v_n = 2n - 1$ for $n = 1, 2, 3, \ldots$

We can define a sequence s by letting $\underline{s_n = u_n + v_n}$ for $n = 1, 2, 3$.

Then s is the sequence

$$2+1,\ 4+3,\ 6+5,\ 8+7,\ 10+9,\ 12+11,\ \ldots$$

that is

$$3,\ 7,\ 11,\ 15,\ 19,\ 23,\ \ldots$$

which is the sequence $s_n = 4n - 1$ for $n = 1, 2, 3, \ldots$

You could see this directly because

$$s_n = u_n + v_n = 2n + (2n - 1) = 4n - 1.$$

You can also create a new sequence by multiplying two known sequences term by term:

EXAMPLE

Let u be the sequence

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \ldots$$

that is $u_n = 2n$ for $n \geq 1$.

Let v be the sequence

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots$$

that is $v_n = 2n-1$ for $n \geq 1$.

We define a sequence p by letting $p_n = u_n \cdot v_n$ for $n = 1, 2, 3, \ldots$
Then p is the sequence

$$2\cdot1,\ 4\cdot3,\ 6\cdot5,\ 8\cdot7,\ 10\cdot9,\ 12\cdot11,\ \ldots$$

that is

$$2,\ 12,\ 30,\ 56,\ 90,\ 132,\ \ldots$$

How can we find a formula for the general term of this sequence?

Well, $p_n = u_n \cdot v_n$ for $n = 1, 2, 3, \ldots$ so

$$\underline{\underline{p_n}} = u_n \cdot v_n = (2n)(2n-1) = \underline{\underline{4n^2 - 2n}}, \quad \text{for } n \geq 1,$$

a formula it would have been quite difficult to guess if we had not known how p was created.

Recursive definitions of sequences

Some sequences are easiest to define recursively, that is, we give the sequence by giving one (or more) initial terms and a recurrence relation giving the (n+1)st term in terms of the previous terms $n, n-1, n-2, \ldots, 2, 1$

EXAMPLE

Let the sequence s be given by $s_0 = 0$ and

$$s_{n+1} = s_n + n \quad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

Then

$s_0 = 0, \quad s_1 = 0 \quad s_2 = 1 \quad s_3 = 3 \quad s_4 = 6 \quad s_5 = 10 \quad s_6 = 15, \ldots$

We say that knowing s_0 and the recurrence relation gives us a way of working out all terms in the sequence.

However, working out s_{1000} means working out all terms $s_{999}, s_{998}, s_{997}, \ldots, s_2$ and s_1 first, which is a lot of work.

EXAMPLE

Let the sequence f be given by $f_0 = 1$ and $f_1 = 1$ and

$$f_{n+1} = (n+1) f_n \qquad \text{for } n = 1, 2, 3, \ldots$$

Then

$f_0 = 1$

$f_1 = 1$

$f_2 = 2 f_1 = 2$

$f_3 = 3 f_2 = 6$

$f_4 = 4 f_3 = 24$

$f_5 = 5 f_4 = 120$

$\vdots$

Do we know this sequence?

"0 factorial"

Recall that $0! = 1$ and $1! = 1$ and $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

$$(n+1)! = (n+1) \overbrace{n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1}^{n!} = (n+1) \cdot n!$$

Conclusion

$f_n = n!$

So we actually could find a formula for f_n in terms of n here.

The Fibonacci Numbers

A very famous sequence is the Fibonacci sequence. It is given by

$$F_1 = 1 \quad \text{and} \quad F_2 = 1$$

$$F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 2$$

$F_1 = 1$

$F_2 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

$F_6 = F_5 + F_4 = 5 + 3 = 8$

$F_7 = F_6 + F_5 = 8 + 5 = 13$

$F_8 = F_7 + F_6 = 13 + 8 = 21$

$F_9 = F_8 + F_7 = 21 + 13 = 34$

$F_{10} = F_9 + F_8 = 34 + 21 = 55$

$\vdots$

It is very difficult to find a formula that gives F_n in terms of n.

Strings

A string over a set X is just a finite sequence of elements from X.

Example

A binary string is a string with symbols from the set $\{0,1\}$.

E.g. 1000101 is a binary string.

The length of a string is the number of elements in it.

E.g. the binary string 1000101 has length 7.

The string with no elements in it is known as the null string and it is denoted λ

Greek letter 'lambda'.

If α and β are strings, the concatenation of α and β,

$$\alpha\beta$$

is just the string consisting of α followed by β.

Example

The concatenation of the binary strings $\alpha = 101$ and $\beta = 110011$ is the binary string

$$\alpha\beta = \underbrace{101}_{\alpha}\underbrace{110011}_{\beta}$$

$\beta\alpha \neq \alpha\beta$

Summation notation

We often have to write the sum of a finite sequence of numbers. An example of this is the sum of the first 100 positive integers which can be written as:

$$\underline{\underline{1 + 2 + 3 + \ldots + 100}}$$

or alternatively as:

$$\underline{\underline{\sum_{r=1}^{100} r}}$$

In general we can represent the sum of the first n terms of a sequence $\{u_r\}$ by:

$$\sum_{r=1}^{n} u_r$$

Example *Summation notation:*

$$\sum_{r=1}^{4} (3r - 1) = 2 + 5 + 8 + 11$$

This is an example of the sum of the sequence defined by

$$u_r = 3r - 1, \qquad \text{for} \qquad 1 \leq r \leq 4.$$

The values 1 and 4 are called the <u>limits</u> of the summation.

We can manipulate *finite sums* in exactly the same way that we can manipulate normal addition. In particular we have the following formulae:

1.

$$\sum_{r=n}^{m}(a_r + b_r) = \sum_{r=n}^{m} a_r + \sum_{r=n}^{m} b_r$$

2.

$$\sum_{r=n}^{m} ca_r = c\sum_{r=n}^{m} a_r$$

Example *If we have*

$$\sum_{r=n}^{m} u_r = 7 \quad \textit{and} \quad \sum_{r=n}^{m} v_r = 15$$

then:

$$\begin{aligned}\sum_{r=n}^{m}(8v_r - 2u_r) &= \sum_{r=n}^{m} 8v_r + \sum_{r=n}^{m} -2u_r \\ &= 8\sum_{r=n}^{m} v_r - 2\sum_{r=n}^{m} u_r \\ &= 8 \times 15 - 2 \times 7 = 120 - 14 = 106.\end{aligned}$$

Before we continue, let us work out the sum

$$\sum_{r=1}^{100} r .$$

Let $S = \sum_{r=1}^{100} r = 1+2+3+4+5+\cdots+100.$

Then

$$S = 1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$$

but also

$$S = 100 + 99 + 98 + 97 + \cdots + 4 + 3 + 2 + 1$$

adding these two gives

$$2S = \underbrace{101 + 101 + 101 + 101 + \cdots + 101 + 101 + 101 + 101}_{100 \text{ terms}} = 100 \times 101$$

but if $2S = 100 \times 101$ then $S = \frac{100 \times 101}{2} = 5050.$

Similarly we can prove

$$\boxed{\sum_{r=1}^{n} r = \frac{n(n+1)}{2}}$$

Example *The sum*

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

This sum can be used to compute sums of finite sequences

$$\{ar+b\}_{r=1}^{n}$$

where a and b are constants like for example

$$\begin{aligned}
\sum_{r=1}^{1000}(2r+1) &= \sum_{r=1}^{1000}(2r) + \sum_{r=1}^{1000} 1 \\
&= 2\sum_{r=1}^{1000} r + \sum_{r=1}^{1000} 1 \\
&= \frac{2\cdot 1000\cdot 1001}{2} + 1000 \\
&= 1000\cdot 1001 + 1000 \\
&= 1001000 + 1000 \\
&= 1002000.
\end{aligned}$$

Example *The sum*

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

can also be used to compute sums of sequences

$$\{ar+b\}_{r=m}^{n}$$

where a and b are constants, but where the first term, s_m, is not necessarily for $m=1$ like for example:

$$\begin{aligned}
\sum_{r=21}^{1000}(2r+1) &= \sum_{r=1}^{1000}(2r+1) - \sum_{r=1}^{20}(2r+1) \\
&= 1002000 - 2\sum_{r=1}^{20} r - \sum_{r=1}^{20} 1 \\
&= 1002000 - 20 \cdot 21 - 20 \\
&= 1002000 - 420 - 20 \\
&= 1002000 - 440 \\
&= 1001560.
\end{aligned}$$

It is often possible to write the same sum in a number of different ways.

Example *Changing variables in a sum:*

$$\sum_{r=1}^{8}(3r+2) = 5+8+11+14+17+20+23+26$$

$$\sum_{s=2}^{9}(3s-1) = 5+8+11+14+17+20+23+26$$

To prove that these are equivalent we use the change of variable given by $s = r + 1$. note $s = r+1 \Leftrightarrow r = s-1$

First change the limits of the sum:

$$r = 1 \Rightarrow s = 2$$

$$r = 8 \Rightarrow s = 9$$

Then change the 'body' of the sum:

$$3r + 2 = 3(s-1) + 2 = 3s - 1$$

Hence

$$\sum_{r=1}^{8}(3r+2) = \sum_{s=2}^{9}(3s-1)$$

EXAMPLE

Compute $\displaystyle\sum_{r=1}^{1000}(r+1)^2 - \sum_{r=5}^{999} r^2$

$$\sum_{r=1}^{1000}(r+1)^2 - \sum_{r=5}^{999} r^2 = \sum_{r=1}^{1000}(r+1)^2 - \sum_{r=1}^{999} r^2 + \sum_{r=1}^{4} r^2$$

$$= \sum_{r=1}^{1000}(r+1)^2 - \sum_{r=1}^{1000} r^2 + 1000^2 + \sum_{r=1}^{4} r^2$$

$$= \sum_{r=1}^{1000}\left((r+1)^2 - r^2\right) \quad + 1000^2 + 1^2 + 2^2 + 3^2 + 4^2$$

$$= \sum_{r=1}^{1000}(r^2 + 2r + 1 - r^2) \quad + \quad 1000000 + 1 + 4 + 9 + 16$$

$$= \sum_{r=1}^{1000}(2r+1) \quad + \quad 1000030$$

$$= 1002000 \quad + 1000030$$

$$= \underline{\underline{2002030}}$$

Product Notation

If we have a product of n numbers

$$a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

we also have a shorthand notation for this, namely

$$\prod_{r=1}^{n} a_r = a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

EXAMPLE

$$10! = \prod_{r=1}^{10} r = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$$

$$\prod_{r=2}^{5} (2r+1)^2 = 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2 = 25 \cdot 49 \cdot 81 \cdot 121$$

The symbol '$\prod$' is a capital Greek letter 'Pi'.

Product notation satisfies the rule

$$\left(\prod_{r=1}^{n} a_r\right)\cdot\left(\prod_{r=1}^{n} b_r\right) = \prod_{r=1}^{n} a_r b_r$$

BUT NOTE CAREFULLY THAT $\prod_{r=1}^{n}(k a_r) \neq k\prod_{r=1}^{n} a_r$

$$\prod_{r=1}^{n}(k\cdot a_r) = k^n\prod_{r=1}^{n} a_r$$

Example

$$\prod_{r=1}^{10}\left(\tfrac{1}{2}\right)^r \cdot \prod_{r=1}^{10} 2^r = \prod_{r=1}^{10}\left(\tfrac{1}{2}\right)^r\cdot 2^r = \prod_{r=1}^{10}\left(\tfrac{1}{2}\cdot 2\right)^r = \prod_{r=1}^{10} 1^r = \prod_{r=1}^{10} 1 = \underline{\underline{1}}$$

$$\prod_{r=1}^{5}(2r) = 2^5\prod_{r=1}^{5} r = 32\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5 = \underline{\underline{32\cdot 120}}$$

Integers and Divisibility

There are two main reasons why any computer scientist needs to study this subject:

- The binary, octal and hexadecimal number systems are used frequently in computing.
- Some important data security algorithms have their roots in the theory of primes and factorisation.

Our main theorem is **The Division Algorithm**:

For all integers a and positive integers b, there exist unique integers q and r such that

alt: $0 \leq r \leq b-1$

$$a = qb + r \text{ where } 0 \leq r < b.$$

Example

$$\overset{a}{100} = \overset{q}{8} \cdot \overset{b}{12} + \overset{r}{4}$$

$$\overset{a}{33} = 3 \cdot \overset{b}{11} + 0$$

$$\overset{a}{27} = 3 \cdot \overset{b}{8} + 3$$

Careful with negative numbers though:

$$-100 = (-9) \cdot 12 + 8$$

Number Systems

Place Value

When we write a number we do so as a sequence of digits.

In the number 1231051 the digit 1 occurs 3 times. On each occurence it has a different value because it is in a different place.

Assume that we are in our normal base 10 number system, then as we move to the left each place has 10 times the value of the previous place. We think of the above number as

$$\begin{aligned} \underline{1}231\underline{0}5\underline{1} &= \underline{1} \times 10^6 + 2 \times 10^5 + 3 \times 10^4 + \underline{1} \times 10^3 \\ &\quad +0 \times 10^2 + 5 \times 10^1 + \underline{1} \times 10^0. \end{aligned}$$

The main consequence of the fact that the remainder on division by a positive integer is unique assures us that this positional system is well-defined.

Consider the number

$$\underline{329} = \underline{3} \times 10^2 + \underline{2} \times 10^1 + \underline{9} \times 10^0$$

There is no other way to write this number in our system because the remainder on division by 10 is unique.

$329 \div 100$	$= \underline{3}$	remainder 29, so write 3 in position 10^2
$29 \div 10$	$= \underline{2}$	remainder 9, so write 2 in position 10^1
$9 \div 10^0$	$= \underline{9}$	remainder 0, so write 9 in position 10^0

Note that we had no choice at any stage as to what to write where.

Other bases

In base 8 each place would have value 8 times the previous place.

Hence in base 8 we would have

$$1231.051 = 1\times8^3+2\times8^2+3\times8^1+1\times8^0+0\times8^{-1}+5\times8^{-2}+1\times8^{-3}.$$

When we write a number we normally work in base 10. If we are in a different base then we normally write the base as a subscript on the right hand side of the number.

We would write 1231.051_8 to signify that we are in base 8.

In base 2 each place would have value 2 times the previous place. Hence in base 2 we would have

$$10101_2 = 1\times2^4+0\times2^3+1\times2^2+0\times2^1+1\times2^0 = 16_{10}+4_{10}+1_{10} = 21_{10}.$$

Suppose that b is some positive integer greater than 1. In base b the number xyz_b means $x \times b^2 + y \times b^1 + z \times b^0$. Hence

$$\begin{aligned} xyz_b \div b &= xy_b && \text{remainder } z \\ xy_b \div b &= x_b && \text{remainder } y \\ x_b \div b &= 0 && \text{remainder } x \end{aligned}$$

So just as in base 10, to represent an integer in base b we need to know the remainders on succesive divisions by b. The division algorithm can be used to calculate these integers.

Example *Convert 3159_{10} into base 8.*

Solution: *We need to calculate the remainders on successive division by 8.*

$$\boxed{\begin{aligned} 3159_{10} \div 8 &= 394_{10} && \textit{remainder } 7 \\ 394_{10} \div 8 &= 49_{10} && \textit{remainder } 2 \\ 49_{10} \div 8 &= 6_{10} && \textit{remainder } 1 \\ 6_{10} \div 8 &= 0 && \textit{remainder } 6 \end{aligned}}$$

Hence we have that $3159_{10} = 6127_8$.

Note: In base b, dividing by b^r, $r \in \mathbb{Z}$, can be thought of as moving the point r places to the left if $r > 0$ and r places to the right if $r < 0$.

Example

$10^2_{10} \cdot 23_{10} = 23\underline{00}_{10}$

$2^2_{10} \times 110_2 = 110\underline{00}_2$

Example *Convert* 56_{10} *into binary (base 2).*

Solution: *We need to calculate the remainders on successive division by 2.*

$$\begin{array}{rcll} 56_{10} \div 2 & = & 28_{10} & \textit{remainder } 0 \\ 28_{10} \div 2 & = & 14_{10} & \textit{remainder } 0 \\ 14_{10} \div 2 & = & 7_{10} & \textit{remainder } 0 \\ 7_{10} \div 2 & = & 3_{10} & \textit{remainder } 1 \\ 3_{10} \div 2 & = & 1_{10} & \textit{remainder } 1 \\ 1_{10} \div 2 & = & 0 & \textit{remainder } 1 \end{array}$$

Hence we have that $56_{10} = 111000_2$.

A Shortcut

If we have a number in binary representation and want to convert into hexadecimal (base 16), then we have to succesively divide by 16. Since $16 = 2^4$ we can achieve this by shifting the 'binary' point 4 places to the left each time we divide.

Example

Convert the binary integer 1011010100 *into base 16.*

Solution: *We group the digits into sets of* 4 *and convert each set of 4 into a hexadeciamal digit.*

$$\underbrace{10}_{2}\underbrace{1101}_{D}\underbrace{0100}_{4} = 2D4_{16}$$

This method will work for converting between bases a and b where b is a power of a. We can of course reverse the method to convert from base b to base a, e.g. from base 16 to base 2 say.