MA014G
Algebra and Discrete Mathematics A

Lecture Notes 5

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LOGIC

Why study logic?

- Logic is the foundation of all mathematics and computing
- to learn how to make correct conclusions
- formal, unambiguous notation helps others to understand exactly what is meant.

The Man and the Portrait

A man was looking at a portrait.
Someone asked, "Whose picture are you looking at?"
The man replied, "Brothers and sisters have I none,
but this man's father is my father's son."

Who is on the picture?

PROPOSITIONS

The statements with which mathematicians work are called propositions.

påståenden

Propositions are either true or false, but not a matter of opinion.

So statements that can be considered true by one person and false by another person are NOT propositions.

EXAMPLE (propositions)

(i) This person is female.

(ii) $x = 3$.

(iii) Sundsvall is in Spain.

(iv) The number 4 is a prime.

(v) The number 4 is not a prime.

EXAMPLE (statements which are not propositions)

(i) My box is heavy.

(ii) The next assignment is difficult

(iii) Go home!

(iv) Shall we go home?

(v) x

(vi) 10

We shall denote propositions by lower case letters, such as

p: Brothers and sisters have I none

q: This man's father is my father's son.

"Simple" statements like these are called primitive propositions.

COMPOUND PROPOSITIONS are combinations of primitive propositions using NOT, AND or OR:

NEGATION (NOT)

p : I am female — (icke p) NOT p : I am not female

q : $x > 5$ — NOT q : $x \not> 5$ (or $x \leq 5$)

NOT p has the opposite truth values to p, that is NOT p is true exactly when p is false.

NOTATION: We write $\neg p$ or $\sim p$ or $\bar{p}$ to mean NOT p.

CONJUNCTION (AND)

p: I am a female $\quad\quad$ q: I am a lecturer

p AND q : I am a female and I am a lecturer.

The conjunction p AND q is true only when BOTH propositions p and q are true.

NOTATION : We write $p \wedge q$ to mean p AND q.

DISJUNCTION (OR)

p: I am a female $\quad\quad$ q: I am a student

p OR q : I am a female or I am a student.

The disjunction p OR q is true when at least one of the propositions p and q is true.

NOTATION: We write $p \vee q$ to mean p OR q.

TRUTH TABLES

A truth table is a way of analysing complicated compound propositions to obtain their truth values, and of establishing equivalences between different propositions.

Any proposition is either true or false.

Let 0 (or F) represent FALSE, and

let 1 (or T) represent TRUE.

The negation $\neg p$ of a proposition p is true precisely when p is false. We describe this in the following table:

p	$\neg p$
0	1
1	0

The truth table for $\neg p$.

The conjunction $p \wedge q$ is true only when <u>both</u> propositions p and q are true.

The truth table for $p \wedge q$ thus looks like this:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

The disjunction $p \vee q$ is true if at least one of the two propositions p and q is true.

The truth table for $p \vee q$ thus looks like this:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Rules for combining AND, OR and NOT

Two compound statements are said to be *equivalent*, if they take on exactly the same truth values for all possible values of the propositions from which they are built.
If p and q are equivalent propositions we write $p \Leftrightarrow q$.

THEOREM

$$(p \wedge q) \vee (p \wedge r) \Leftrightarrow p \wedge (q \vee r)$$

$$(p \vee q) \wedge (p \vee r) \Leftrightarrow p \vee (q \wedge r)$$

$$\text{NOT}(p \wedge q) \Leftrightarrow (\text{NOT}\, p) \vee (\text{NOT}\, q)$$

De Morgan's Laws.

$$\text{NOT}(p \vee q) \Leftrightarrow (\text{NOT}\, p) \wedge (\text{NOT}\, q)$$

a proposition which is always true is called a tautology,

a proposition which is always false is called a contradiction.

motsägelse

EXAMPLE

$(\neg p) \wedge p$ is a contradiction

$(\neg p) \vee p$ is a tautology

as can be seen from the truth tables:

p	$\neg p$	$(\neg p) \wedge p$
0	1	0
1	0	0

p	$\neg p$	$(\neg p) \vee p$
0	1	1
1	0	1

CONDITIONAL PROPOSITIONS

In mathematics we often have compound propositions of the type

IF p THEN q

where p and q are primitive propositions.

EXAMPLE Consider the following proposition about integers:

(Cm) (sã)

IF $\underbrace{x > 2}_{p}$ THEN $\underbrace{x^2 > 4}_{q}$.

This proposition is true <u>for any value of x.</u>

The integers can be divided into three disjoint subsets

$\{\ldots, -5, -4, -3\}$ $\{-2, -1, 0, 1, 2\}$ $\{3, 4, 5, 6, \ldots\}$

Let us compute the truth value for p and q for an x in each of these:

	$p: x > 2$	$q: x^2 > 4$	IF p THEN q
$x \in \{\ldots, -5, -4, -3\}$	0	1	1
$x \in \{-2, -1, 0, 1, 2\}$	0	0	1
$x \in \{3, 4, 5, \ldots\}$	1	1	1

CONCLUSION: The proposition IF p THEN q is ONLY FALSE

WHEN <u>p is true and q is false.</u>

Notation

There are many ways in which to express the proposition

IF p THEN q.

In symbols we write this as $p \rightarrow q$ or $p \Rightarrow q$

or even $q \leftarrow p$ or $q \Leftarrow p$.

In words we say

(i) IF p THEN q	(i) OM p, SÅ q
(ii) q if p	(ii) q om p
(iii) p only if q	(iii) p endast om q
(iv) p implies q	(iv) p implicerar (el. medför) q
(v) p is a sufficient condition for q	(v) p är ett tillräckligt villkor för q.
( ) q is a necessary condition for p	(vi) q är ett nödvändigt villkor för p.

We concluded above that $p \Rightarrow q$ is only false when p is TRUE and q is FALSE. Thus the truth table for $p \Rightarrow q$ looks as follows:

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

This has important implications for mathematical proofs and arguments:

* If p is true and we argue correctly, we can NEVER reach a false conclusion.

* However, if p is false, we can obtain either a true or a false conclusion.

EXAMPLE

$1 = 0 \Rightarrow 1+1 = 1+0 \Rightarrow 2 = 1$ "false $\Rightarrow$ false"

$1 = 0 \Rightarrow 1 \cdot 0 = 0 \cdot 0 \Rightarrow 0 = 0$ "false $\Rightarrow$ true"

The case of the stupid defence lawyer

In a land where logic rules life,

a man has been accused of robbery and is in court.

The following statements are made by the prosecutor and the defence lawyer.

prosecutor: IF the defendant (the man accused) is guilty
THEN he had an accomplice (medbrottsling).

defence : That is <u>not</u> true!

Why is this the worst possible thing the defence lawyer could have said?

A bad proof of a correct theorem

<u>Exercise</u>: Prove for any number x that $x^2-5+1 = (x-2)(x+2)$

<u>One student wrote</u>: <u>BEVIS</u>:

$$
\begin{aligned}
& x^2-5+1 \overset{?}{=} (x-2)(x+2) \\
\Downarrow\ & \\
& x^2-4 = x^2-2x+2x-4 \\
\Downarrow\ & \\
& x^2-4 = x^2-4 \\
\Downarrow\ & \\
& 0 = 0
\end{aligned}
$$

för att den sista raden är sann, så är den första raden, dvs. $x^2-5+1=(x-2)(x+2)$ sann. V.S.V.

Why did the lecturer not accept this proof?

How does a good proof for this identity look?

Let us return to the truth table for $p \Rightarrow q$:

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

<u>p IF AND ONLY IF q</u> (p om och endast om q)

This statement is true exactly when both $p \Rightarrow q$ and $q \Rightarrow p$ are true, that is when p and q have the same truth value.

We write $p \Leftrightarrow q$ or p iff q (or p omm q in Swedish).

We have the following truth table

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

QUANTIFIERS

Mathematical propositions often contain the words "for all" or "there exists", for example

(a) For all integers n, $n+1$ is an integer.

(b) There exists an integer n such that $n^2 = 0$.

(c) Every natural number is positive.

(d) There is an integer between 3 and 5.

We use the symbol $\forall$ as shorthand-notation for "for all"

and we use symbol $\exists$ as shorthand-notation for "there exists"

We can then write the above propositions:

(a) $\forall\, n \in \mathbb{Z} : (n+1) \in \mathbb{Z}$

(b) $\exists\, n \in \mathbb{Z} : n^2 = 0$

(c) $\forall\, n \in \mathbb{N} : n > 0$

(d) $\exists\, x \in \mathbb{Z} : 3 < x < 5$

When negating propositions containing quantifiers $\forall$ and $\exists$, these get exchanged for each other

$p : \forall\, n \in \mathbb{Z} : n^2 = 0$ $\qquad$ $\neg p : \exists\, n \in \mathbb{Z} : n^2 \neq 0$

$q : \exists\, x \in \mathbb{Z} : 3 < x < 5$ $\qquad$ $\neg q : \forall\, x \in \mathbb{Z} \quad x \leq 3$ or $x \geq 5$

PROOF

The concept of proof is a vital part of mathematics; it allows us to demonstrate that statements are true. In general we cannot do this by simply checking that they work for a few, or even many, values; it may be correct for these and yet fail for others.

We shall look at three methods of proof. These are:

1. Direct proof
2. Proof by contradiction
3. Proof by Induction

Before we can start proving a proposition we must make sure that whatever we talk about is well-defined, that is all simple propositions can be understood by every reader unambiguously and every proposition must make sense in the situation we are in. Consider the following:

Definition An immovable post is a post which can not be knocked over by anything.
An irresistible cannonball is a cannonball which knocks over everything in its path.

Is this a good definition?

Question: What happens if an irresistible cannonball hits an immovable post?

① Direct Proof

To prove : $p \Rightarrow q$

Method : We start by assuming <u>p is true</u> and by a sequence of logical implications end up with <u>q is true</u>

Reason : $p \Rightarrow q$ is only false when <u>p is true and q is false</u>. By arguing that <u>p is true</u> implies <u>q is true</u>, we have eliminated this case.

EXAMPLES

- Prove that there exist integers a, b and c such that $a^2 + b^2 = c^2$.

- Prove that if x is an even integer then x^2 is an even integer.

We saw earlier on p. 117 that there are many ways of expressing the proposition $p \Rightarrow q$ (p IMPLIES q)

There is one more equivalent way of expressing $p \Rightarrow q$, namely

$$\underline{\text{NOT } q \Rightarrow \text{NOT } p}$$

This is known as the contrapositive statement of $p \Rightarrow q$.

It can be proved by truth tables that the two statements are equivalent, but we shall not do so on this course.

However, it gives us one more way of proving $p \Rightarrow q$ directly, namely:

Contrapositive proof

To prove: $p \Rightarrow q$

Method: Prove "NOT $q \Rightarrow$ NOT p".

Reason: "$p \Rightarrow q$" and "NOT $q \Rightarrow$ NOT p" are logically equivalent.

EXAMPLE

For all integers n,

IF n^2 is even THEN n is even.

② Proof by contradiction

To prove : $p \Rightarrow q$

Method : Assume that (p AND not q) is true and obtain a contradiction (something false).

Reason : If we start with a proposition and argue correctly to obtain something false, we must have started with a false proposition
So if we start by $(p \wedge \neg q)$ and reach a contradiction, then $(p \wedge \neg q)$ is false, so $\neg(p \wedge \neg q)$ is true, and thus by the following truth table we have proved $p \Rightarrow q$:

p	q	$p \Rightarrow q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	0
1	1	1	0	1

EXAMPLES

- The smallest factor greater than 1 of any integer $n > 1$ is prime.
- $\sqrt{2}$ is an irrational number
- Prove for all $x, y \in \mathbb{R}$ that if $x + y \geq 1000$ then $x \geq 500$ or $y \geq 500$.

Proving biconditional statements

To prove : $p \Leftrightarrow q$

Method : Prove both $p \Rightarrow q$ and $q \Rightarrow p$.
(sometimes we need not split it, but can prove $p \Leftrightarrow q$ directly, we shall see an example of this later)

Reason : By definition $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is equivalent to $p \Leftrightarrow q$.

EXAMPLES

For all integers n, n^2 is even $\Leftrightarrow$ n is even.

$x^2 - 4 = 5$ if and only if $x = 3$ or $x = -3$.

Proof by counterexample

To prove : The proposition $\forall x, p(x)$ is false.

Method : Find an x where $p(x)$ is false (counterexample)

Reason : If there exists an x for which $p(x)$ is false, then the proposition $\forall x, p(x)$ is false.

EXAMPLES

(i) Everyone in this room has grey eyes.

(ii) Everyone in this room is asleep.

(iii) $\forall x \in \mathbb{R}, x^2 \geq x$.

(iv) $\forall n \in \mathbb{N}$ $2^{2^n}+1$ is prime

counterexample to (iv) ?

$n=1$ $2^{2^1}+1 = 2^2+1 = 5$ prime

$n=2$ $2^{2^2}+1 = 2^4+1 = 17$ prime

$n=3$ $2^{2^3}+1 = 2^8+1 = 257$ prime

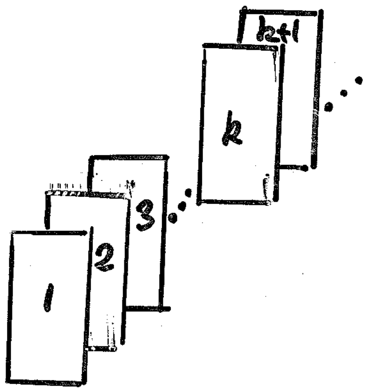
$n=4$ $2^{2^4}+1 = 2^{16}+1 = 65537$ prime

$n=5$ $2^{2^5}+1 = 2^{32}+1 = 4\,294\,967\,297 = 641 \times 6\,700\,417$ not prime (counter-example)

$\vdots$

③ Proof by induction

The idea:



"domino-effect"

If domino 1 is knocked down then domino 2 falls and then 2 knocks down 3 and so on...

That is, if domino k falls, then domino $k+1$ falls too

Suppose we know the following two things:

(i) Domino number 1 falls

(ii) Domino k falls $\Rightarrow$ Domino $k+1$ falls

Conclusion: Domino n falls for all $n \geq 1$.

If we know:

(i) Domino 7 falls

(ii) Domino k falls $\Rightarrow$ Domino $k+1$ falls

Conclusion: Domino n falls for all $n \geq 7$.

Let $p(n)$ denote the proposition "domino n falls", then

IF

(i) $p(1)$ is true

(ii) $p(k) \Rightarrow p(k+1)$ for any integer $k \geq 1$

THEN $p(n)$ is true for all integers $n \geq 1$

More general,

IF

(i) $p(b)$ is true (for some <u>fixed</u> base-number b)

(ii) $p(k) \Rightarrow p(k+1)$ for any integer $k \geq b$

THEN $p(n)$ is true for all integers $n \geq b$.

<u>EXAMPLES</u>

① $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$ for $n \geq 1$

② $n! \geq 2^n$ for $n \geq 4$.

③ Let a number sequence be given by

$$u_{n+1} = u_n + (n+1)^2 \quad \text{for } n = 0, 1, \ldots$$

and $u_0 = 0$.

Show that $u_n = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \geq 0$.

QUESTION: Is n^2+n+41 a prime for all integers $n \geq 0$?

Primes 2 - 3571

2	233	547	877	1229	1597	1993	2371	2749	3187
3	239	557	881	1231	1601	1997	2377	2753	3191
5	241	563	883	1237	1607	1999	2381	2767	3203
7	251	569	887	1249	1609	2003	2383	2777	3209
11	~~257~~	571	907	1259	1613	2011	2389	2789	3217
13	263	577	911	1277	1619	2017	2393	2791	3221
17	269	587	919	1279	1621	2027	2399	2797	3229
19	271	593	929	1283	1627	2029	2411	2801	3251
23	277	599	937	1289	1637	2039	2417	2803	3253
29	281	601	941	1291	1657	2053	2423	2819	3257
31	283	607	947	1297	1663	2063	2437	2833	3259
37	293	613	953	1301	1667	2069	2441	2837	3271
41	307	617	967	1303	1669	2081	2447	2843	3299
43	311	619	971	1307	1693	2083	2459	2851	3301
47	313	631	977	1319	1697	2087	2467	2857	3307
53	317	641	983	1321	1699	2089	2473	2861	3313
59	331	643	991	1327	1709	2099	2477	2879	3319
61	337	647	997	1361	1721	2111	2503	2887	3323
67	347	653	1009	1367	1723	2113	2521	2897	3329
71	349	659	1013	1373	1733	2129	2531	2903	3331
73	353	661	1019	1381	1741	2131	2539	2909	3343
79	359	673	1021	1399	1747	2137	2543	2917	3347
83	367	677	1031	1409	1753	2141	2549	2927	3359
89	373	683	1033	1423	1759	2143	2551	2939	3361
97	379	691	1039	1427	1777	2153	2557	2953	3371
101	383	701	1049	1429	1783	2161	2579	2957	3373
103	389	709	1051	1433	1787	2179	2591	2963	3389
107	397	719	1061	1439	1789	2203	2593	2969	3391
109	401	727	1063	1447	1801	2207	2609	2971	3407
113	409	733	1069	1451	1811	2213	2617	2999	3413
127	419	739	1087	1453	1823	2221	2621	3001	3433
131	421	743	1091	1459	1831	2237	2633	3011	3449
137	431	751	1093	1471	1847	2239	2647	3019	3457
139	433	757	1097	1481	1861	2243	2657	3023	3461
149	439	761	1103	1483	1867	2251	2659	3037	3463
151	443	769	1109	1487	1871	2267	2663	3041	3467
157	449	773	1117	1489	1873	2269	2671	3049	3469
163	457	787	1123	1493	1877	2273	2677	3061	3491
167	461	797	1129	1499	1879	2281	2683	3067	3499
173	463	809	1151	1511	1889	2287	2687	3079	3511
179	467	811	1153	1523	1901	2293	2689	3083	3517
181	479	821	1163	1531	1907	2297	2693	3089	3527
191	487	823	1171	1543	1913	2309	2699	3109	3529
193	491	827	1181	1549	1931	2311	2707	3119	3533
197	499	829	1187	1553	1933	2333	2711	3121	3539
199	503	839	1193	1559	1949	2339	2713	3137	3541
211	509	853	1201	1567	1951	2341	2719	3163	3547
223	521	857	1213	1571	1973	2347	2729	3167	3557
227	523	859	1217	1579	1979	2351	2731	3169	3559
229	541	863	1223	1583	1987	2357	2741	3181	3571

EXAMPLE

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

for all $n=1,2,3,\ldots$

PROOF

Base case $(n=\underline{1})$

$LHS = 1$

$RHS = \frac{1\cdot(1+1)}{2} = \frac{1\cdot 2}{2} = 1$

So $LHS=RHS$ for $n=1$.

Inductive hypothesis (Swedish: för något $k \geq 1$)

Assume $1+2+\cdots+k = \frac{k(k+1)}{2}$ for some $k \geq 1$.

base

Induction step

We must now prove $1+2+\cdots+k+k+1 = \frac{(k+1)(k+1+1)}{2}$ $\circledast$

But

$$\begin{aligned}
\text{LHS of } \circledast &= 1+2+\cdots+k+1 = (1+2+\cdots+k)+k+1 \\
&= \frac{k(k+1)}{2} + (k+1) \quad \text{by the inductive hypothesis} \\
&= (k+1)\left(\frac{k}{2}+1\right) \\
&= (k+1)\left(\frac{k+2}{2}\right) \\
&= \frac{(k+1)(k+2)}{2} \\
&= \text{RHS of } \circledast
\end{aligned}$$

So the result holds for $n=k+1$ also and therefore holds for all $n \geq 1$ by induction. ■

Example 3.12 *Example of a proof by induction:*

Prove by induction that for all $n \geq 4$, $2^n < n!$.

Base Case: $n = 4$. $2^4 = 16$ *and* $4! = 4 \times 3 \times 2 \times 1 = 24$. *Therefore* $2^4 < 4!$ *as required.*

Inductive hypothesis: *The result is true for some* $n = k \geq 4$ *that is* $2^k < k!$.

Inductive step: *We need to prove that this is true for* $n = k + 1$, *i.e. that* $2^{k+1} < (k+1)!$

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&> (k+1)2^k \ \textit{(by the inductive hypothesis)} \\
&> 2 \times 2^k \ \textit{(as } k > 4 \textit{ and thus } k+1 > 2\textit{)} \\
&= 2^{(k+1)}
\end{aligned}$$

Therefore $(k+1)! > 2^{(k+1)}$, *and so the result holds by the principle of induction.*

EXAMPLE

The sequence given by

$$u_{n+1} = u_n + (n+1)^2 \quad \text{for } n=0,1,\ldots$$

and $u_0 = 0$ has the general term

$$u_n = \frac{n(n+1)(2n+1)}{6} \quad \text{for all } n \geq 0.$$

PROOF

base case $(n=0)$

$u_0 = 0$ and $\frac{0\cdot(0+1)(2\cdot 0+1)}{6} = 0$ so for $n=0$: $u_n = \frac{n(n+1)(2n+1)}{6}$

Inductive hypothesis

(Swedish: för något $k \geq 0$)

Assume $u_k = \frac{k(k+1)(2k+1)}{6}$ for some $k \geq 0$

Induction step

We must prove $u_{k+1} = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$. ⊛

But

LHS of ⊛ $= u_{k+1} = u_k + (k+1)^2$ by the recurrence relation

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by the inductive hypothesis}$$

$$= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right)$$

$$= (k+1)\left(\frac{2k^2+k+6k+6}{6}\right) = (k+1)\left(\frac{2k^2+7k+6}{6}\right)$$

$$= \frac{(k+1)(2k+3)(k+2)}{6} = \text{RHS of ⊛}$$

So the result holds for $n=k+1$ and thus for all $n \geq 0$ by induction.

Recurrence relations

Recall from block 2 that a recurrence relation is a way of defining a sequence $\{a_n\}_{n=0}^{\infty}$ by expressing the n'th term a_n in terms of its predecessors $a_{n-1}, a_{n-2}, \dots, a_1, a_0$ together with some initial terms 'to get the recursion started'.

Examples

(1) The Fibonacci sequence is given by

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2$$

and initial conditions $a_0 = 1$, $a_1 = 1$.

(2) The factorial sequence is given by

$$f_n = n f_{n-1} \quad \text{for } n \geq 1$$

and initial condition $a_0 = 1$.

Question: Can we find a closed expression for the n'th term a_n, i.e. one that does not involve other terms of the sequence?

This is called to **SOLVE the recurrence relation.**

Some recurrence relations are easy to solve. You can

(1) Guess a solution and then

(2) Prove by induction that your guess is correct.

EXAMPLE

Let $a_n = 2a_{n-1}$ for $n \geq 1$ and $a_0 = 1$

(1) Analysis: $a_n = 2a_{n-1} = 2(2a_{n-2}) = 2^2 a_{n-2}$

$$= 2^2(2a_{n-3}) = 2^3 a_{n-3}$$

$$= 2^3(2a_{n-4}) = 2^4 a_{n-4} \ldots$$

Guess: $a_n = 2^n$

(2) Prove $a_n = 2^n$ for $n \geq 0$ by induction!

We shall learn how to solve a special kind of recurrence relation called a linear homogeneous recurrence relation with constant coefficients. It is of the type

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are constant real numbers.
The recurrence relation is said to be of order r if $c_r \neq 0$ and $c_{r+1} = 0 = c_{r+2} = \cdots = c_k$.

EXAMPLES

Which of the following recurrence relations are linear, homogeneous with constant coefficients?

1. The Fibonacci sequence: $a_n = a_{n-1} + a_{n-2}$

2. The factorial sequence: $a_n = n\, a_{n-1}$

3. $a_n = 2a_{n-1}$

4. $a_n = a_n a_{n-1}$

5. $a_n = a_{n-1} + 2$

6. $a_n = a_{n-1}^2$

7. $a_n = -a_{n-1} + 2a_{n-2} - 3a_{n-3}$

Solving linear homogeneous recurrence relations with constant coefficients:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad \text{for } n \geq k$$

where $c_1, c_2, \ldots, c_k$ are constants, with initial terms $a_0, a_1, \ldots, a_{k-1}$ given.

① Substitute $a_i = t^i$ $\quad i = 0, 1, \ldots, n$ into the r.r.:

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_{n-k} t^{n-k}$$
$$\Downarrow$$
$$t^k = c_1 t^{k-1} + c_2 t^{k-2} + \cdots + c_{n-k} t^0$$
$$\Downarrow$$
$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \cdots - c_{n-k} = 0$$

This is known as the characteristic equation of the rec. rel.

② Find the k roots of the characteristic equation

ⓐ If all k roots are distinct numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, then the solution of the rec. rel. is of the form

$$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_k \alpha_k^n$$

where $A_1, A_2, \ldots, A_k$ are constants

ⓑ If some roots are repeated roots, e.g. if $\alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_m}$ then the terms in the solution for this repeated root will be

$$A_{i_1} \alpha_{i_1}^n + A_{i_2} \cdot \underset{\substack{\uparrow \\ \text{OBS!}}}{n} \cdot \alpha_{i_1}^n + A_{i_3} \underset{\substack{\uparrow \\ \text{OBS!}}}{n^2} \alpha_{i_1}^n + \cdots + A_{i_m} \underset{\substack{\uparrow \\ \text{OBS!}}}{n^{m-1}} \alpha_{i_1}^n$$

in the solution.

③ Determine $A_1, A_2, \ldots A_k$ from the initial terms.

EXAMPLE

Solve the recurrence relation

$$a_n = 3a_{n-1} + 4a_{n-2} \quad \text{for } n \geq 2$$

with initial conditions $a_0 = a_1 = 1$

① $t^n = 3t^{n-1} + 4t^{n-2}$

$\Downarrow$

$t^2 = 3t + 4$

$\Downarrow$

$t^2 - 3t - 4 = 0$ (characteristic equation)

② Roots of the characteristic equation are $\dfrac{3 \pm \sqrt{9+16}}{2} = \begin{cases} 4 \\ -1 \end{cases}$

So $\alpha_1 = 4$ and $\alpha_2 = -1$.

ⓐ Hence the general solution to the recurrence relation is of the form

$$a_n = A_1 4^n + A_2(-1)^n$$

③ $a_0 = 1$ so $1 = A_1 4^0 + A_2(-1)^0$

$\Downarrow$

$\underline{A_1 + A_2 = 1}$

$a_1 = 1$ so $\underline{1 = 4A_1 - A_2}$

So $5A_1 = 2 \Rightarrow A_1 = \frac{2}{5} \Rightarrow A_2 = \frac{3}{5}$

④ The solution is

$$\underline{\underline{a_n = \tfrac{2}{5} 4^n + \tfrac{3}{5}(-1)^n}} \quad \text{for } n \geq 0$$

EXAMPLE

Solve the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} \quad \text{for } n \geq 2,$$

with initial conditions $a_0 = a_1 = 1$.

① $t^n = 6t^{n-1} - 9t^{n-2}$

$\Downarrow$

$t^2 = 6t - 9$

$\Downarrow$

$t^2 - 6t + 9 = 0$ (characteristic equation)

② Roots of the characteristic equation are $\frac{6 \pm \sqrt{36-36}}{2} = 3$

So $\alpha_1 = 3$ and $\alpha_2 = 3$, i.e. 3 is a root of multiplicity 2.

ⓑ Hence the general solution to the recurrence relation is of the form

$$a_n = A_1 3^n + A_2 n 3^n$$

③ $a_0 = 1$ so $1 = A_1 3^0 + A_2 \cdot 0 \cdot 3^0$

$\Downarrow$

$A_1 = 1$

$a_1 = 1$ so $1 = 1 \cdot 3 + A_2 \cdot 1 \cdot 3$

$\Downarrow$

$A_2 = \frac{-2}{3}$

④ The solution is

$$a_n = 3^n - \tfrac{2}{3} n 3^n = \underline{\underline{(3-2n)\,3^{n-1}}} \quad \text{for } n \geq 0$$

EXAMPLE

Solve the recurrence relation

$$a_n = -a_{n-2} \quad \text{for } n \geq 2$$

with initial conditions $a_0 = 0$, $a_1 = 1$.

① $t^n = -t^{n-2}$

$\Downarrow$

$t^2 = -1$

$\Downarrow$

$t^2 + 1 = 0$ (characteristic equation)

② Roots of the characteristic equation are i and $-i$,

So $\alpha_1 = i$ and $\alpha_2 = -i$.

ⓐ Hence the general solution of the r.r. is of the form

$$a_n = A_1 i^n + A_2 (-i)^n.$$

③ $a_0 = 0$ so $\underline{0 = A_1 + A_2}$

$a_1 = 1$ so $\underline{1 = A_1 i + A_2(-i)}$

$\Downarrow$

$$0 = A_1 i + A_2 i$$
$$1 = A_1 i + A_2 (-i)$$

$\Downarrow$

$$1 = 2iA_1$$

$\Downarrow$

$$A_1 = \frac{1}{2i}, \quad A_2 = -\frac{1}{2i}$$

④ The solution is for all $n \geq 0$: $a_n = \frac{1}{2i} i^n - \frac{1}{2i}(-i)^n = \underline{\underline{\frac{1}{2}\left(i^{n-1} + (-i)^{n-1}\right)}}$