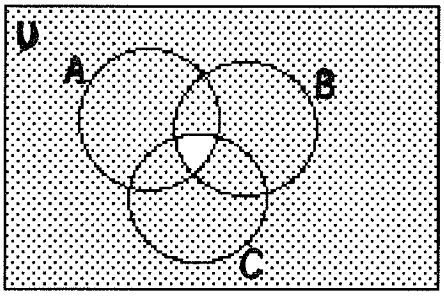
Uppgift 1

(a) (i) & (ii) $X = \overline{(A \cap B \cap C)}$ is the shaded area in the following Venn diagram.



(iii) Use the Associative Laws and De Morgan's Law twice:

$$\overline{(A \cap B \cap C)} = \overline{A \cap (B \cap C)} = \overline{A} \cup \overline{(B \cap C)} = \overline{A} \cup (\overline{B} \cup \overline{C}) = \overline{A} \cup \overline{B} \cup \overline{C}.$$

(b) $\{2^r + 1 : \ r \in \mathbb{Z} \ \text{ and } \ -3 \leq r \leq 3\} = \{\frac{9}{8}, \frac{5}{4}, \frac{3}{2}, 2, 3, 5, 9\}$.

(c) $|X| = 4$ so $|\mathcal{P}(X)| = 2^4 = 16$. Hence, by the Multiplication Principle

$$|Y \times \mathcal{P}(X)| = |Y||\mathcal{P}(X)| = 2 \cdot 16 = 32.$$

Comments:
In (a)(iii) you could also as an alternative solution have made a Venn diagram depicting the RHS, compared this with the Venn diagram from (ii) and explained why they show the two sets are equal.
In (b) make sure you remember the braces.
In (c) it would be acceptable to list the set $Y \times \mathcal{P}(X)$ and count the number of elements, but in order to get full marks, you would need to get the set absolutely right with all braces etc.

Uppgift 2

(a) (i) $(10100100001110)_2 = (10\,1001\,0000\,1110)_2 = (290E)_{16}$;

(ii) $(290E)_{16} = 2 \cdot 16^3 + 9 \cdot 16^2 + 0 \cdot 16 + 14 = 10510$.

(b) (i) $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$;

(ii) $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) $\displaystyle\sum_{n=1}^{50} 4n = 4\sum_{n=1}^{50} n = 4 \cdot \frac{50 \cdot 51}{2} = 100 \cdot 51 = 5100$;

$$\begin{aligned}
\sum_{n=3}^{50} (2n-1)^2 &= \sum_{n=1}^{50} (2n-1)^2 - 1^2 - 3^2 = \sum_{n=1}^{50} (4n^2 - 4n + 1) - 10 \\
&= 4\sum_{n=1}^{50} n^2 - \sum_{n=1}^{50} 4n + \sum_{n=1}^{50} 1 - 10 \\
&= 4 \cdot \frac{50 \cdot 51 \cdot 101}{6} - 5100 + 50 - 10 = 166640.
\end{aligned}$$

Comments:

In (a) all working must be shown, otherwise it will result in the loss of most of the marks. I have used the shortcut method for converting from binary to hex in (a)(i), but you may of course also use the longer standard method of converting to base 10 first and then doing successive divisions by 16 to get hex. Similarly, in (a)(ii) you could have used the binary number to find the base 10 number rather than converting the hexadecimal from (i).

In (b) both formulas must be correct in order to obtain the marks. Check that your index used matches the variable in the 'body' of the sum and that n is not used ambiguously (n must not be both a limit and the index of the sum).

In (c) make sure you show all the steps. Do not attempt to solve an exercise like this by computing all 48 terms and adding them on the calculator, it will gain you little or no points as the exercise clearly specifies that the formulas must be used to obtain the result.

Uppgift 3

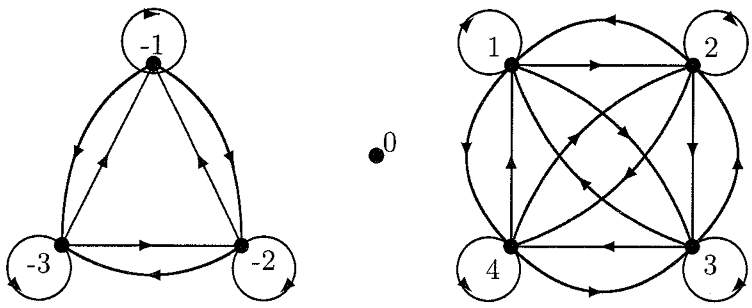
(a) (i) R is reflexive if xRx for all $x \in S$;

(ii) R is symmetric if $xRy \Rightarrow yRx$ for all $x, y \in S$;

(iii) R is transitive if xRy and $yRz \Rightarrow xRz$ for all $x, y, z \in S$;

(iv) R is antisymmetric if xRy and $yRx \Rightarrow x = y$ for all $x, y \in S$.

(b) (i) The digraph of R :



(ii) R is **not reflexive** as 0 is not related to 0 under R. R is **symmetric** as $xy = yx$, so $s_1Rs_2 \Rightarrow s_2Rs_1$. R is **transitive** as $xy > 0$ iff x and y are non-zero and have the same sign, hence if s_1Rs_2 and s_2Rs_3 then also s_1Rs_3 as s_1 and s_3 both have the same sign as s_2. R is **not antisymmetric** as e.g. $1R2$ and $2R1$ but $1 \neq 2$.

(iii) R is not an equivalence relation as it is not reflexive;

(iv) R is not a partial order as it is not reflexive.

(c) (i) The domain of f is A, The codomain of f is $\mathbb{Z}$ and the range of f is $\{-1, 1, 3, 5, 7\}$.

(ii) f is not onto as the codomain and the range of f are not the same.

(iii) f is one-to-one, for if $f(x_1) = f(x_2)$ then $2x_1 - 3 = 2x_2 - 3$ and thus $x_1 = x_2$.

(iv) f is not invertible as f is not one-to-one.

Comments:

Definitions like the ones in (a) must be stated carefully using the right symbols. For example, forgetting 'for all...' in one or more of the definitions, any wrong use of '$\Rightarrow$' and wrong use of the words 'and', 'or' and 'implies' and confusing the names of properties will result in the loss of marks.

In (b)(ii)-(iv) half of the marks are for spotting the correct properties and the other half for some explanation, but not necessarily quite as rigorous as mine.

In (c)(iii) a less formal explanation as to why f is one-to-one would be acceptable.

Uppgift 4

(a) $u_1 = 2u_0 + 2^0 = 2 \cdot 1 + 1 = 3,$
$u_2 = 2u_1 + 2^1 = 2 \cdot 3 + 2 = 8,$
$u_3 = 2u_2 + 2^2 = 2 \cdot 8 + 4 = 20,$
$u_4 = 2u_3 + 2^3 = 2 \cdot 20 + 8 = 48,$
$u_5 = 2u_4 + 2^4 = 2 \cdot 48 + 16 = 112,$
$u_6 = 2u_5 + 2^5 = 2 \cdot 112 + 32 = 256.$

(b) We prove by induction that

$$u_n = 2^{n-1}(n+2) \text{ för alla } n \geq 0.$$

Base case: $LHS = u_0 = 1$ and $RHS = 2^{0-1}(0+2) = 2^{-1}2 = 1$. So the result holds for $n = 0$.

Inductive Hypothesis: Assume $u_k = 2^{k-1}(k+2)$ for some $k \geq 0$.

Inductive Step: We must prove that $u_{k+1} = 2^k(k+3)$:

$$\begin{aligned} u_{k+1} &= 2u_k + 2^k \text{ by the recurrence relation} \\ &= 2 \cdot 2^{k-1}(k+2) + 2^k \text{ by the inductive hypothesis} \\ &= 2^k(k+2) + 2^k \\ &= 2^k(k+2+1) \\ &= 2^k(k+3), \end{aligned}$$

whence the result holds for $n = k+1$ and thus for all $n \geq 0$ by induction.

Comments:
In (a) make sure you demonstrate that you have used the recurrence relation and *not* the formula in (b) to calculate u_1 to u_6 here, otherwise you will lose all marks. In the proof by induction, do not forget the 'for some $k \geq 0$'-part of the hypothesis, make sure you indicate where the inductive hypothesis is used in the inductive step, make sure you indicate where the recurrence relation is used in the inductive step, do not forget the concluding line of the proof, and show all computations carefully. Make sure you *know* for sure two things are equal before you put down an ' ='-sign between them.

Uppgift 5

(a) (i) The vertices in the graph are going to be labelled by the towns. Two vertices are adjacent if there exists a direct bus connection between the two corresponding towns. The graph is not going to have loops because the bus connections are all from one town to another. It will not have multiple edges because the graph only depicts whether there exists a direct connection, not how many there are. Hence it is simple.

(ii) The number of pairs of towns with a direct bus connection is the number of edges in the graph. The table gives all the degrees of the vertices, so by the Handshaking Lemma there are $(3+3+2+2+3+3)/2 = 8$ edges in the graph and thus 8 pairs of towns have a direct bus connection.

(b) No, because such a graph would have $(2+2+2+3+3+3)/2 = 7.5$ edges by the Handshaking Lemma, and this cannot be, as 7.5 is not an integer.

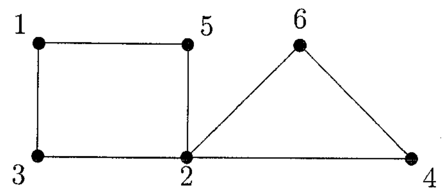
(c) (i) Two simple graphs are said to be isomorphic if there exists a one-to-one correspondence of their vertex sets which preserves adjacency.

(ii) H_1 and H_2 are not isomorphic as e.g. the number of 3-cycles in H_1 is 2 while the number of 3-cycles in H_2 is just 1.

Comments:
Note that in (a) you cannot draw the graph from the information given, as there are more than one graph satisfying the given conditions, so you cannot know how the graph is going to look. Hence in (a)(ii) you *must* demonstrate that you have used the Handshaking Lemma and not just counted the edges in one example of a graph satisfying the conditions. In (c)(i) try to explain the definition of an isomorphism in your own words .
In (c)(ii) the marks are for any correct argument why the two graphs are not isomorphic, mine is just one example. In order to prove two graphs are non-isomorphic you need to point out some structure in one graph which does not exist in the other, it is not enough to attempt to set up an isomorphism and tell that you failed.

Uppgift 6

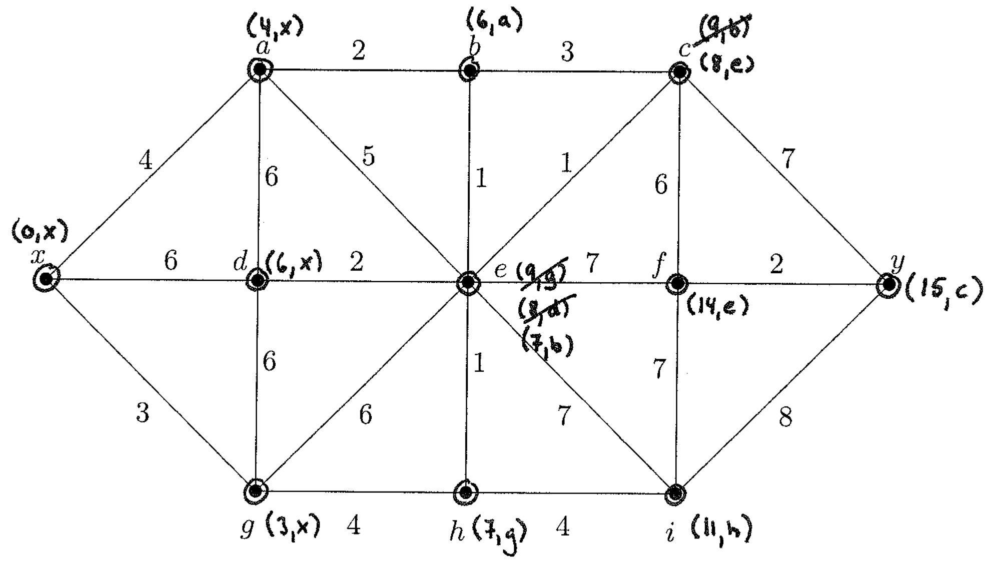
(a) G is the graph



G is not bipartite as it contains a triangle, so the vertices cannot be partitioned (2-coloured) into two sets such that adjacent vertices always are in opposite sets. G is Eulerian as all vertices have even degree (The cycle $(4, 2, 3, 1, 5, 2, 6, 4)$ is an Euler cycle). G is not Hamiltonian as e.g. no cycle can contain both vertices 5 and 6 without passing through 2 twice, hence there can be no simple cycle containing all vertices precisely once.

(b) The shortest path from x to y in the graph below is (x, a, b, e, c, y), it has weight 15.
The labels on the vertices of the graph below are the ones attached to the vertices by Dijkstra's algorithm. The sequence in which the vertices become marked by the algorithm is one of the following 4:

$$x, g, a, d/b, b/d, e/h, h/e, c, i, f, y.$$



Comments:
In (a) half the marks are for finding the correct properties of the graph, the rest is for the arguments why the graph has these properties/does not have them. In (b) the answer carries next to no value, the crucial thing is to show that you have understood how to use Dijkstra's algorithm and that you have used it. You can label the vertices as I have done here or make a table as in Sarah Norell's note about shortest path which shows how the vertices get labelled and in which order they have been chosen.

Uppgift 7

(a) For all integers a and positive integers n there exist *unique* integers q and r (called the quotient and the remainder) such that $a = qn + r$ and $0 \leq r < n$.

(b) (i) $q = 11$ and $r = 1$; (ii) $q = -12$ and $r = 8$.

(c) (i) $a \equiv b \pmod{n}$ if $n|(a-b)$.

(ii) Suppose that a gives remainder r_a and b gives remainder r_b on division by n according to the Division algorithm. Then there are integers q_a and q_b such that

$$\begin{aligned} a &= q_a n + r_a \text{ where } 0 \leq r_a < n \text{ and} \\ b &= q_b n + r_b \text{ where } 0 \leq r_b < n. \end{aligned}$$

Subtracting the two equations yields

$$a - b = (q_a - q_b)n + (r_a - r_b)$$

which can be rewritten as

$$r_a - r_b = (a-b) - (q_a - q_b)n,$$

and here n divides the RHS as $a \equiv b \pmod{n}$, hence it also divides $r_a - r_b$. However $-n < r_a - r_b < n$, so since n divides it, $r_a - r_b = 0$ and so $r_a = r_b$ as required.

(d) The five congruence classes of the (equivalence) relation $a \equiv b \pmod 5$ on $\mathbb{Z}$ are $[i]_5 = \{x \in \mathbb{Z} : x \equiv i \pmod 5\}$ where $i = 0, 1, 2, 3, 4$.

$$\mathbb{Z}_5 = \{[0]_5, [1]_5, [2]_5, [3]_5, [4]_5\}.$$

The multiplication and addition tables for $\mathbb{Z}_5$ are

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

if you omit brackets and subscripts such that i denotes the class $[i]_5$.

Uppgift 8

(a)

$$\begin{aligned}\mathbf{114} &= \mathbf{72}\cdot 1+\mathbf{42}\\ \mathbf{72} &= \mathbf{42}\cdot 1+\mathbf{30}\\ \mathbf{42} &= \mathbf{30}\cdot 1+\mathbf{12}\\ \mathbf{30} &= \mathbf{12}\cdot 2+\mathbf{6}\\ \mathbf{12} &= \mathbf{6}\cdot 2+\mathbf{0}.\end{aligned}$$

Hence $\gcd(114,72)=6=\mathbf{30}-\mathbf{12}\cdot 2=\mathbf{30}-\mathbf{42}-\mathbf{30})\cdot 2=\mathbf{30}\cdot 3-\mathbf{42}\cdot 2=(\mathbf{72}-\mathbf{42})\cdot 3-\mathbf{42}\cdot 2=\mathbf{72}\cdot 3+\mathbf{42}(-5)=\mathbf{72}\cdot 3+(\mathbf{114}-\mathbf{72})(-5)=\mathbf{72}\cdot 8+\mathbf{114}(-5)$

(b) $\gcd(114,72)=6$ and $6 \not| 2$, hence $72x\equiv 2 \pmod{114}$ has no solutions.

(c) We solve

$$72x\equiv 6 \pmod{114}$$

which has the same integer solutions as

$$12x\equiv 1 \pmod{19}$$

which in $\mathbb{Z}_{19}$ is equivalent to the equation

$$[12]\odot[x]=[1].$$

But from (a) we have $6=72\cdot 8+114(-5)$ whence $1=12\cdot 8+19(-5)$ such that $[8]$ is the multiplicative inverse of $[12]$ in $\mathbb{Z}_{19}$. Multiply on both sides with this inverse then $[x]=[8]=\{\dots,-30,-11,8,27,\dots\}$ are all the solutions.

(d) We found above that the integer solutions x to the congruence $72x\equiv 6 \pmod{114}$ are $x\in[8]_{19}$, hence $[x]\in\mathbb{Z}_{114}$ is a solution iff $x\in[8]_{19}$. But then

$$[x]=[8]_{114},[27]_{114},[46]_{114},[65]_{114},[84]_{114} \text{ or } [103]_{114}$$

as all the numbers in these classes are congruent to 8 modulo 19.

Comments:
Notice how Euclid's algorithm was used to find the inverse of $[12]$ in $\mathbb{Z}_{19}$. Most of the marks in this question are for being able to find this inverse and for doing the computation in Euclid's algorithm forwards and backwards.

Uppgift 9

(a) A k-combination from a set with n elements is an unordered selection (i.e. a subset) with k elements from the set with n elements.

(b) First count the number of *ordered* lists of k elements from the set of n elements. When choosing the first element in the list there are n choices, for the second element in the list there are $n-1$ choices, ..., for the kth element there are $n-(k-1)$ choices. Thus, using the Multiplication Principle there are

$$n \cdot (n-1) \cdot \ldots \cdot (n-(k-1)) = \frac{n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1}{(n-k) \cdot (n-(k+1)) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-k)!}$$

ordered lists of k elements from the set of n elements. When counting these, we have counted each unordered selection $k!$ times as there are $k!$ ways of ordering a set with k elements. The total number of unordered selections is thus $\frac{n!}{k!(n-k)!}$.

(c) There is a one-to-one correspondence between unordered selections of size k picked from a set with n elements and the unordered selections of size $n-k$ not picked, hence $\binom{n}{k} = \binom{n}{n-k}$.

(d) (i) **A combinatorial argument.** Say we want to choose a subset with k elements from a set X with $n+1$ elements. The number of ways of doing this is given by the LHS, so we need to explain why the RHS also counts this. We partition the selections into two types:

- the selections where we do *not* include the first element of the set X, of which there are $\binom{n}{k}$;
- the selections where we *do* include the first element of the set X; these consist of the first element together with $k-1$ of the n other elements of X, hence there are $1 \cdot \binom{n}{k-1} = \binom{n}{k-1}$ of these.

Adding the two yields $\binom{n}{k}+\binom{n}{k-1}$ which is then the total number of subsets of size k from the set X of size $n+1$.

(ii) **Verifying the identity algebraically:**

$$\begin{aligned}
RHS = \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} \\
&= \frac{n!(n-(k-1))}{k!(n-(k-1))(n-k)!} + \frac{n!k}{k(k-1)!(n-(k-1))!} \\
&= \frac{n!(n+1-k)}{k!(n-(k-1))!} + \frac{n!k}{k!(n-(k-1))!} \\
&= \frac{n!(n+1-k)+n!k}{k!(n-(k-1))!} \\
&= \frac{n!(n+1-k+k)}{k!(n-(k-1))!} = \frac{n!(n+1)}{k!(n-(k-1))!} = \frac{(n+1)!}{k!(n-(k-1))!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k} = LHS.
\end{aligned}$$

Comments:

In (c) you could also have verified the identity algebraically using (b).
The idea with this optional exercise is to test that you have a good understanding of combinations and that you have understood the formula for the binomial coefficient $\binom{n}{k}$ rather than just memorising it. The optional exercise will either have a strong theoretical content or contain harder computations than the other exercises.