MID SWEDEN UNIVERSITY

TFM

Examination 2005

MAAA99 Algebra and Discrete Mathematics (English)

Time: 5 hours

Date: 2 November 2005

There are EIGHT questions on this paper and each question carries three points. The maximum number of points available is 24. The points for each part of a question are indicated at the end of the part in [ ]-brackets. Ten points are needed for the mark G and eighteen points are required for the mark VG.

The candidates are advised that they must always show their working, otherwise they will not be awarded full marks for their answers.

The candidates are further advised to start each of the eight questions on a new *page and to clearly label all their answers.*

This is a closed book examination. No books, notes or mobile telephones are allowed in the examination room.

Electronic calculators may be used provided they cannot handle formulas. The make and model used must be specified on the cover of your script.

GOOD LUCK!!

Make sure that you are doing the right Discrete Mathematics A paper! This paper is for the course which used

[Johnsonbaugh]

as coursebook.

Question 1

(a) Let A,B and C be subsets of the universal set $\mathcal{U}$.

(i) Shade the area $X = A - (B \cap C)$ in a Venn diagram.

(ii) Shade the area $Y = (A \cup B) - (B \cup C)$ in a Venn diagram.

(iii) Showing all your working, decide whether $X = Y$ for *every* possible choice of A, B and C.

(iv) Find three non-empty sets of positive integers A, B and C such that

$$A - (B \cap C) = (A \cup B) - (B \cup C).$$

[2p]

(b) Describe the set

$$M_1 = \{\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{5}{5}, \tfrac{7}{5}, \tfrac{9}{5}, \ldots\}$$

by using the rules of inclusion method. [0.5p]

(c) Describe the set

$$M_2 = \{x \in \mathbb{N} \mid x^2 + 7x + 12 = 0\}$$

by the listing method. [0.5p]

Question 2

For each of the following, decide which options are correct and which are incorrect. Give a short explanation for each of your answers.

(a) The number of 7-bit binary strings containing exactly 4 zeros is

(i) 35

(ii) $2^4 \cdot 1 \cdot 1 \cdot 1$

(iii) 7^4

(iv) 4^7

(v) $7 \cdot 6 \cdot 5 \cdot 4$ [0.75p]

(b) A function $f : \{1, 2, 3, 4\} \to \{a, b, c\}$

(i) is never one-to-one

(ii) is always one-to-one

(iii) is never onto

(iv) is always onto [0.75p]

(c) The sum $10 + 20 + 30 + \ldots + 1000$ can also be written as

(i) $\displaystyle\sum_{i=0}^{99}(1000 - 10i)$

(ii) $\displaystyle\sum_{i=1}^{100} 10n$

(iii) $\displaystyle 10\sum_{n=1}^{100} n$

(iv) 50500 [0.75p]

(d) Let n be an integer. The proposition

$$\text{‘if } n^2 - 9 < 0 \text{ then } -3 < n < 3\text{’}$$

is the contrapositive of the proposition

(i) ‘if $n^2 - 9 \geq 0$ then $-3 \geq n$ or $n \geq 3$’

(ii) ‘if $-3 \geq n$ and $n \geq 3$ then $n^2 - 9 \geq 0$’

(iii) ‘$n^2 - 9 \geq 0$ if $-3 \geq n$ or $n \geq 3$’

(iv) ‘if $3 > n > -3$ then $n^2 - 9 \geq 0$’ [0.75p]

Question 3

(a) Explain what is meant by saying that a relation R on a set S is

(i) reflexive;

(ii) symmetric;

(iii) transitive;

(iv) anti-symmetric;

(v) a partial order. [0.5p]

(b) Justifying your answer, give a relation $\mathcal{R}$ which is a partial order on the set of integers $\mathbb{Z}$. [1p]

(c) Let R be the relation on the set $S = \{-3, -2, -1, 1, 2, 3, 4\}$ defined by

$$(s_1, s_2) \in R \text{ iff } s_1 s_2 > 0.$$

(i) Draw the relation digraph of R.

(ii) Show that the relation R is reflexive, symmetric and transitive.

(iii) Show that the relation R is an equivalence relation.

(iv) List the equivalence classes of R on S. [1.5p]

Question 4

A sequence $\{u_n\}$ is given by the recurrence relation

$$u_{n+1} = u_n + 2n \quad \text{for } n = 1, 2, 3, \ldots,$$

and the initial term $u_1 = 1$.

(a) Showing all your working, use the recurrence relation to compute the terms u_2, u_3, u_4 and u_5. [1p]

(b) Prove by induction that

$$u_n = n^2 - n + 1 \quad \text{for all} \quad n \geq 1.$$

[2p]

Question 5

(a) In a tennis tournament there are 6 competitors where each competitor plays every other precisely once. Explain how such a tournament can be modelled by a graph. [0.5p]

(b) Alice and Bob are two spectators who both keep scores. Alice decides to count the wins of each competitor, while Bob counts their losses. Here are their results:.

	Alice (no. of wins)	Bob (no. of losses)
Player 1	2	3
Player 2	2	3
Player 3	4	1
Player 4	3	2
Player 5	3	1
Player 6	1	4

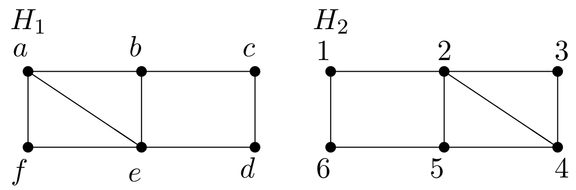
(i) Explain why Bob has got the scores wrong.

(ii) Justifying your answer, say whether Alice has got them right.

(iii) If Alice has got the scores right and Bob has made just one error, correct Bob's error in the table. [1p]

(c) (i) When are two simple graphs said to be *isomorphic*?

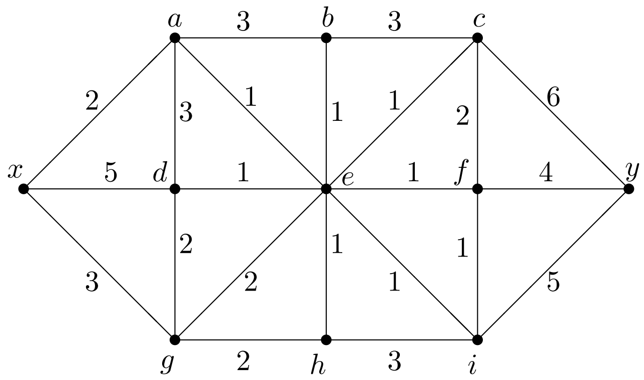
(ii) Show that the graphs H_1 and H_2 below are isomorphic. [1.5p]



Question 6

(a) Use Kruskal's or Prim's algorithm to find a minimal spanning tree for the weighted graph below. Show your working, i.e. show in which order the algorithm adds the edges to the spanning tree.
Further, give the total weight of the minimal spanning tree. [1.5p]

(b) Showing all your working, use *Dijkstra's algorithm* to find a shortest path from vertex x to vertex y in the weighted graph below. Take care to show how the vertices become labelled and how the labels change during the run of the algorithm. Finally, give the length of the shortest path. [1.5p]



Question 7

(a) (i) What is meant by saying that $a \equiv b \pmod 4$?

(ii) List the 4 congruence classes of the congruence relation $a \equiv b \pmod 4$ on $\mathbb{Z}$.

(iii) Describe the set $\mathbb{Z}_4$.

(iv) Compute the addition table for $\mathbb{Z}_4$.

(v) Compute the multiplication table for $\mathbb{Z}_4$.

(vi) List all the elements of $\mathbb{Z}_4$ which have a multiplicative inverse. [1.5p]

(b) (i) Let $n \geq 2$ be an integer. State a result which helps you decide whether an element $[x] \in \mathbb{Z}_n$ has a multiplicative inverse or not.

(ii) Use the result you stated in (i) to find an element $[a] \in \mathbb{Z}_{120}$ such that $[a] \neq [0]$ but still $[a]$ has *no* multiplicative inverse.

(iii) Find two elements $[x], [y] \in \mathbb{Z}_{120}$ such that $[x] \neq [y]$, $[x] \neq [0]$ and $[y] \neq [0]$ but

$$[3] \odot [x] = [3] \odot [y].$$

[1.5p]

Question 8

(a) (i) Let a and b be two positive integers.
State the conditions a positive integer g must satisfy in order for g to be called $\gcd(a, b)$, *the greatest common divisor* of a and b.

(ii) Use *Euclid's algorithm* to show that $\gcd(3571, 1753) = 1$.

(iii) Find two integers s and t such that

$$1753s + 3571t = 1.$$

[1.75p]

(b) Showing all your working, find all solutions $[x] \in \mathbb{Z}_{3571}$ to the equation

$$[1753] \odot [x] = [10].$$

[0.75p]

(c) Showing all your working, find the set of all integers x which satisfy the congruence

(i) $1753x \equiv 2 \pmod{3571}$;

(ii) $3506x \equiv 4 \pmod{7142}$. [0.5p]