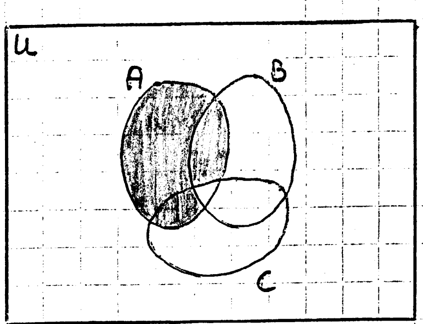
MAAA99 2 NOVEMBER 2005

Q1

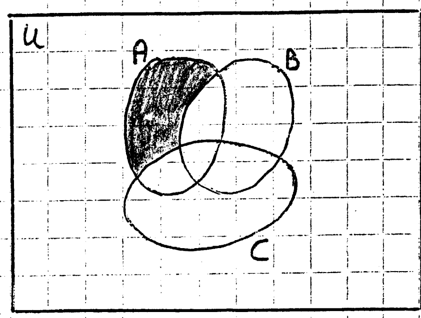
(a)(i)



$X = A - (B \cap C)$ is shaded.

2

(a)(ii)



$Y = (A \cup B) - (B \cup C)$ is shaded

2

(a)(iii) In general $X \neq Y$ as the two Venn diagrams above are different. 2

(a)(iv) E.g. $A = \{1,2\}$ $B = \{2,3\}$ $C = \{2,4\}$

then $A - (B \cap C) = \{1,2\} - \{2\} = \{1\}$ 2

and $(A \cup B) - (B \cup C) = \{1,2,3\} - \{2,3,4\} = \{1\}$

(b) $M_1 = \{ \frac{2n+1}{5} \mid n \in \mathbb{N} \}$ 2

(c) $M_2 = \{ \}$ 2

12

Q2

(a) $\binom{7}{4} = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot 2 \cdot 1} = 35$, so (i) is the right answer. 3

(b) f cannot be one-to-one as $\{1,2,3,4\}$ has higher cardinality than $\{a,b,c\}$, so (i) is correct.

(ii) is thus wrong.

(iii) is wrong as e.g. $f(1)=a$, $f(2)=b$, $f(3)=c$, $f(4)=c$ would be onto.

(iv) is wrong as e.g. $f(1)=f(2)=f(3)=f(4)=a$ would not be onto.

3

(c)

(i) is correct as $\sum_{i=0}^{99}(1000-10i) = 1000+990+980+\cdots+10$

(ii) is wrong as $\sum_{i=1}^{100} 10n = \underbrace{10n+10n+\cdots+10n}_{100 \text{ terms}} = 1000n$

(iii) is correct as $10\sum_{n=1}^{100} n = 10(1+2+3+\cdots+100) = 10+20+\cdots+1000$

(iv) is correct as $10\sum_{n=1}^{100} n = 10 \cdot \frac{100 \cdot 101}{2} = 5 \cdot 10100 = 50500$

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(d) (iii) is correct as the contrapositive of $p \Rightarrow q$ is not $q \Rightarrow$ not p.

3

12

Q.3

(a) R is reflexive if xRx $\forall x \in S$.

R is symmetric if $xRy \Rightarrow yRx$ $\forall x, y \in S$

R is transitive if $xRy, yRz \Rightarrow xRz$ $\forall x, y, z \in S$

R is anti-symmetric if $xRy, yRx \Rightarrow x = y$ $\forall x, y \in S$

R is a partial order if R is reflexive, anti-symmetric and transitive

2

(b) E.g. '$\leq$' is a partial order on $\mathbb{Z}$,

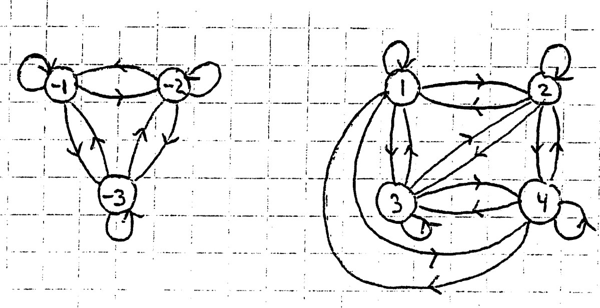
it is reflexive as $x \leq x$ $\forall x \in \mathbb{Z}$,

it is anti-symmetric as $x \leq y, y \leq x \Rightarrow x = y$ $\forall x, y \in \mathbb{Z}$,

it is transitive as $x \leq y, y \leq z \Rightarrow x \leq z$ $\forall x, y, z \in \mathbb{Z}$

4

(c) (i)



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(ii) R is reflexive as $x^2 > 0$ $\forall x \in S$ $\frac{1}{2}$

R is symmetric, for if xRy then $xy > 0$ and so x and y have the same sign, whence $yx > 0$ also. Thus yRx. 1

R is transitive, for if xRy, yRz then x, y have the same sign and y, z have the same sign. Thus x, z have the same sign $\Rightarrow xz > 0 \Rightarrow xRz$. 1

(iii) As R is reflexive, symmetric and transitive, R is an equivalence relation $\frac{1}{2}$

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(iv) $\{-1, -2, -3\}$, $\{1, 2, 3, 4\}$ are the two equivalence classes of R on S. 1

Q4

(a) $u_1 = 1$

$u_2 = u_1 + 2\cdot 1 = 1+2 = 3$

$u_3 = u_2 + 2\cdot 2 = 3+4 = 7$ 4

$u_4 = u_3 + 2\cdot 3 = 7+6 = 13$

$u_5 = u_4 + 2\cdot 4 = 13+8 = 21$

(b) Base $u_1 = 1$ and $1^2 - 1 + 1 = 1$ so result holds for $n=1$. 1

Ind. hyp. Assume $u_k = k^2 - k + 1$ for some $k \geq 1$. 1 1

Ind. step We must prove $u_{k+1} = (k+1)^2 - (k+1) + 1$.

$LHS = u_{k+1} = u_k + 2k$ by the recurrence relation

$= k^2 - k + 1 + 2k$ by the inductive hypothesis.

$= k^2 + k + 1$

$RHS = (k+1)^2 - (k+1) + 1$ 5

$= k(k+1) + 1$

$= k^2 + k + 1$

$= LHS$ ✓

so the result holds for $n = k+1$ and thus for all $n \geq 1$ by induction.

12

Q5

(a) A complete graph K_6 with directed edges can be used to model such a tournament. The vertices will be the players and an arc $a \to b$ means player a beat player b. 2

(b) (i) $\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ matches are being played, Bob's sum is 14. 1

(ii) Alice's sum is 15, but if she made more than one mistake, the two mistakes may cancel out each other so that the final sum is still 15. Hence we cannot know whether she got it right. 2

(iii) In K_6 each vertex has degree 5 (corresponding to each player playing 5 matches). Hence, if Alice got it right, Bob's mistake is for player 5 which should have 2 losses. 1

(c) (i) Two simple graphs G and H are isomorphic if there is a 1-1 correspondence φ of their vertex sets such that if (x,y) is an edge of G then $(\varphi(x), \varphi(y))$ is an edge of H.

2

(ii)

x	a	b	c	d	e	f
$\varphi(x)$	4	5	6	1	2	3

is an isomorphism 2

as it carries edges to edges:

$(a,b) \to (4,5)$
$(b,c) \to (5,6)$ 2
$(c,d) \to (6,1)$
$(d,e) \to (1,2)$
$(e,f) \to (2,3)$
$(f,a) \to (3,4)$
$(a,e) \to (4,2)$
$(b,e) \to (5,2)$

12

Q6

(a) Using Kruskal's algorithm the edges chosen are

(a, e) wt 1 1

(d, e) wt 1

(b, e) wt 1 1

(h, e) wt 1 1

(f, i) wt 1

(c, e) wt 1

(e, i) wt 1

(x, a) wt 2

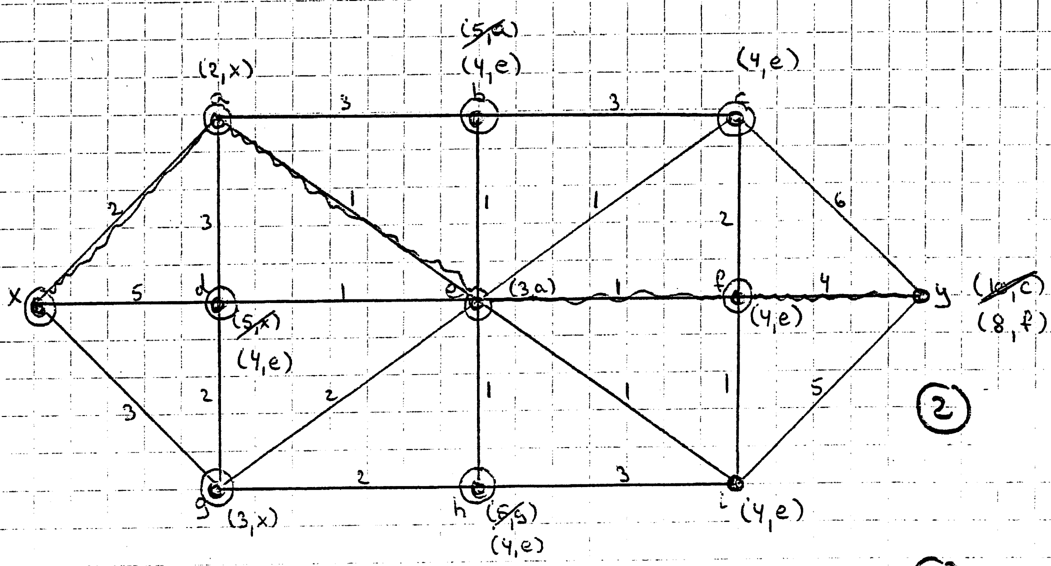
(g, h) wt 2

(f, y) wt 4 (at which point the algorithm stops as we have chosen 2
10 edges and there are just 11 vertices in the graph.)

15

Total weight of MST is 15. 1 ~~for total weight.~~

(b)



Vertices chosen in the order: x, a, g/e, e/g, $\underbrace{d, b, c, h, f, i}_{\text{in any order}}$, y 2

So the shortest path is (x, a, e, f, y) of length 8 2

12

Q7

(a) (i) $a \equiv b \pmod 4$ if $4 \mid (a-b)$ 1

(ii)

$[0]_4 = \{4k \mid k \in \mathbb{Z}\}$

$[1]_4 = \{4k+1 \mid k \in \mathbb{Z}\}$ 1

$[2]_4 = \{4k+2 \mid k \in \mathbb{Z}\}$

$[3]_4 = \{4k+3 \mid k \in \mathbb{Z}\}$

(iii) $\mathbb{Z}_4 = \{[0]_4, [1]_4, [2]_4, [3]_4\}$ 1

(iv)+(v) omitting square brackets and subscripts$_4$, the tables are

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\odot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2

(vi) From the multiplication table we see that only $[1]_4$ and $[3]_4$ have mult. inv. 1

(b) (i) $[x] \in \mathbb{Z}_n$ has a multiplicative inverse iff $\gcd(x,n) = 1$. 2

(ii) e.g. $[40] \in \mathbb{Z}_{120}$ has no mult. inv. as $\gcd(40,120) = 40 \neq 1$. 2

(iii) e.g. $[3]_{120} \odot [40]_{120} = [3]_{120} \odot [80]_{120}$ (both $= [0]_{120}$)

while $[40]_{120} \neq [80]_{120}$ 2

$\boxed{12}$

Q 8

(a)(i) • $g|a$, $g|b$

1

• if $d|a$ and $d|b$ then $d|g$

(ii)
$$\begin{aligned}
\underline{3571} &= \underline{1753} \times 2 + \underline{65} \\
\underline{1753} &= \underline{65} \times 26 + \underline{63} \\
\underline{65} &= \underline{63} \times 1 + \underline{2} \\
\underline{63} &= \underline{2} \times 31 + \underline{1} \\
\underline{2} &= \underline{1} \times 2 + \underline{0}
\end{aligned}$$

3

so $\gcd(3571, 1753) = 1 = 63 - 2 \cdot 31 = 63 - (65-63) \cdot 31 = 63 \cdot 32 + 65(-31)$

$= (1753 - 65 \cdot 26) \cdot 32 + 65(-31) = 1753(32) + 65(-863)$

$= 1753(32) + (3571 - 1753 \cdot 2)(-863) = 1753(1758) + 3571(-863)$

(iii) so let $s = 1758$ and $t = -863$ then $1753s + 3571t = 1$.

3

(b)(i) $\gcd(1753, 3571) = 1$ so there is one solution in $\mathbb{Z}_{3571}$

By (a)(iii) the mult inv of $[1753]_{3571}$ is $[1758]_{3571}$, so multiplying by that on both sides yields the solution $[x] = [17580]_{3571} = [3296]_{3571}$

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(c)(i) From (b) the mult. inv. of $[1753]_{3571}$ is $[1758]_{3571}$ so $x \equiv 1758 \cdot 2 \pmod{3571}$

$\Updownarrow$

$x \equiv 3516 \pmod{3571}$

so $x \in [3516]_{3571} = \{3516 + k \cdot 3571 \mid k \in \mathbb{Z}\}$

1

(ii) $3506x \equiv 4 \pmod{7142} \iff 1753x \equiv 2 \pmod{3571}$

so (ii) has the same set of solⁿˢ as (i).

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12