

MID SWEDEN UNIVERSITY

TFM

Examination 2006

MAAA99 Algebra and Discrete Mathematics (English)

Time: 5 hours

Date: 10 January 2006

There are EIGHT questions on this paper and each question carries three points. The maximum number of points available is 24. The points for each part of a question are indicated at the end of the part in []-brackets. Ten points are needed for the mark G and eighteen points are required for the mark VG.

The candidates are advised that they must always show their working, otherwise they will not be awarded full marks for their answers.

The candidates are further advised to start each of the eight questions on a new page and to clearly label all their answers.

This is a closed book examination. No books, notes or mobile telephones are allowed in the examination room.

Electronic calculators may be used provided they cannot handle formulas. The make and model used must be specified on the cover of your script.

GOOD LUCK!!

Make sure that you are doing the right Discrete Mathematics A paper! This paper is for the course which used

[Johnsonbaugh]

as coursebook.

Question 1

- (a) Write down the negation of the proposition:

‘Each student went to at least one New Year’s party this year.’ [1p]

- (b) Convert the decimal integer 2006 to base 8. [0.5p]

- (c) (i) Find $a \in \{0, 1, 2, \dots, 12\}$ such that $2006 \equiv a \pmod{13}$;
(ii) Find $b \in \{0, 1, 2, \dots, 12\}$ such that $2006^2 \equiv b \pmod{13}$
and check that $b \equiv a^2 \pmod{13}$;
(iii) Find $x \in \{0, 1, 2, \dots, 12\}$ such that $2006^{666} \equiv x \pmod{13}$. [1.5p]

Question 2

- (a) Algot and Georg are in the sport shop to buy clothing for the boys football team. Algot buys socks costing 16 kronor a pair and Georg buys T-shirts at 46 kronor each. They shop for precisely 618 kronor altogether and they buy more pairs of socks than T-shirts. How many pairs of socks and how many T-shirts do they buy? *Justify your answer, an answer found by trial and error gives at most half a mark!* [1.5p]

- (b) Another day Algot wanted to buy a new computer, but he had no money. He had n old cars though which he sold for n thousand kronor each. For the money from the sale of the cars he first purchased as many motorcycles as possible at 24 thousand kronor each. When the motorcycles had been paid for, he still had more than 12 thousand kronor left, and it turned out that he had precisely money enough to buy the new computer. What was the price of the computer?
Hint: Use modular arithmetic. The answer is actually uniquely determined by the information given though n and the number of motorcycles are not! [1.5p]

Question 3

- (a) Explain what is meant by saying that a relation R on a set S is
- (i) reflexive;
 - (ii) symmetric;
 - (iii) transitive;
 - (iv) an equivalence relation. [0.5p]
- (b) Justifying your answer, decide for each of the following two relations on the set of integers $\mathbb{Z}$, whether it is an equivalence relation or not. If you find that a relation is an equivalence relation, list its equivalence classes.
- (i) $(x, y) \in R_1$ iff $7|xy$;
 - (ii) $(x, y) \in R_2$ iff $x^2 = y^2$. [2.5p]

Question 4

A sequence $\{u_n\}$ is given by the linear homogeneous recurrence relation

$$2u_n - 7u_{n-1} + 3u_{n-2} = 0 \quad \text{for } n = 2, 3, 4, \dots,$$

and the initial terms $u_0 = 7$ and $u_1 = 1$.

- (a) Showing all your working, use the recurrence relation to compute the terms u_2, u_3, u_4 and u_5 . [1p]
- (b) Solve the recurrence relation. [2p]

Question 5

Let the sum s_n be defined by

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}$$

for all positive integers n .

(a) Express s_n using Σ -notation. [1p]

(b) Prove by induction that

$$s_n = \frac{n}{n+1}$$

for all $n \geq 1$. [2p]

Question 6

Santa visits the Svensson family on Christmas Eve. There are 7 children in the family, but two of them have not been quite as good this year as they could have been, so Santa has only 5 presents for the family. All the presents are different, but unfortunately Santa has forgotten his list, so he does not know which present goes to which child, and he cannot remember which two children have been naughty.

(a) In how many ways can Santa distribute the 5 presents among the 7 children, if all presents are to be given away and each child gets at most one present. [1.5p]

Santa is a nice man, and he feels sorry for the two Svensson children who did not get a present, so he invites them for a ride in his sleigh. After this all the children are very happy and they all dance around the Christmas tree.

(b) In how many ways can the 7 children arrange themselves in a circle to dance around the Christmas tree?

[1.5p]

Question 7

- (a) Consider the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by the rule

$$f(x) = 7x - 2.$$

- (i) Give the *domain*, and the *range* of f .
- (ii) Prove that f is *one-to-one*.
- (iii) Prove that f is *not onto*.

[1.5p]

- (b) Consider the function $g : \mathbb{Z} \rightarrow \mathbb{Z}$ given by the rule

$$g(x) = ax - b,$$

where a and b are integers.

Find all pairs (a, b) for which g is *invertible*.

[1.5p]

Question 8

- (a) The trapper Algy Matheson must check his traps in the forest. The network of paths and distances between the traps (in kilometres) is shown in the graph below. The snow is very deep in the forest and his snowshoes are old and broken, so it is hard work to walk along a new path, while the effort of walking back along a path he has already walked is negligible. He must visit all his traps. What is the minimum total length of paths with new snow he must walk to be able to do so? Justify your answer!

[1.5p]

- (b) The next day, starting from trap a , Algy must visit his traps again and does a breadth-first traversal along the paths he has opened up. In which order does he visit the traps?

[1.5p]

