**MID SWEDEN UNIVERSITY**

**TFM**

**Examination 2006**

## MAAA99 Algebra and Discrete Mathematics (English)

**Time: 5 hours**

**Date: 5 June 2006**

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*There are EIGHT questions on this paper and each question carries three points. The maximum number of points available is 24. The points for each part of a question are indicated at the end of the part in [ ]-brackets. Ten points are needed for the mark G and eighteen points are required for the mark VG.*

*The candidates are advised that they must always show their working, otherwise they will not be awarded full marks for their answers.*

*The candidates are further advised to start each of the eight questions on a new page and to clearly label all their answers.*

**This is a closed book examination. No books, notes or mobile telephones are allowed in the examination room. Note that a collection of formulas is attached to the paper.**

**Electronic calculators may be used provided they cannot handle formulas. The make and model used must be specified on the cover of your script.**

*GOOD LUCK!!*

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**Make sure that you are doing the right Discrete Mathematics A paper! This paper is for the course which used**

**[Johnsonbaugh]**

**as coursebook.**

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**Question 1**

(a) Express the binary number $(11011100001001010)_2$

(i) as a hexadecimal;

(ii) in base 10. [1p]

(b) Write down the following formula by using $\Sigma-$notation.

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

[1p]

(c) Compute the following sum by using the formula from (b).

$$\sum_{n=20}^{200} (2n+1)^2.$$

[1p]

**Question 2**

The sequence $\{s_n\}_{n=1}^{\infty}$ is defined by

$$s_n = \sum_{r=0}^{n} (r+1)^2 \quad \text{for } n \geq 1.$$

(a) Showing all our working, compute $s_1, s_2, s_3, s_4, s_5$ and $s_6$. [1p]

(b) Prove by induction that

$$s_n = \frac{(n+1)(n+2)(2n+3)}{6}$$

for all $n \geq 1$. [2p]

**Question 3**

(a) Let $A$,$B$ and $C$ be subsets of the universal set $\mathcal{U}$.

(i) Shade the region $X = (A \cup B) \cap \overline{C}$ in a Venn diagram.

(ii) Let $Y = A \cup (B \cap \overline{C})$. Justifying your answer, say whether $X = Y$ for *all* possible choices of $A$, $B$ and $C$.

(iii) Find three non-empty subsets $A$, $B$ and $C$ of the positive integers such that

$$(A \cup B) \cap \overline{C} = A \cup (B \cap \overline{C}).$$

[1.5p]

(b) Describe the set

$$M_1 = \{0, -\tfrac{2}{3}, \tfrac{4}{5}, -\tfrac{6}{7}, \tfrac{8}{9}, -\tfrac{10}{11}, \ldots\}$$

by using the rules of inclusion method.

[0.5p]

(c) Describe the following two sets by the listing method.

(i) $M_2 = \{x \in \mathbb{R} \mid 5x^2 + 2x - 3 = 0\}$;

(ii) $\mathcal{P}(M_2 \cup \{0\})$. [1p]

**Question 4**

(a) Explain what it means for a relation $R$ on a set $S$ to be

(i) reflexive;

(ii) symmetric;

(iii) transitive;

(iv) anti-symmetric. [1p]

(b) Let $R$ be the following relation on the set $S = \{w, x, y, z\}$.

$$R = \{(w,w), \quad (w,y), \quad (x,w), \quad (x,x), \quad (y,w), \quad (y,z), \quad (z,w)\}$$

(i) Draw the relation digraph of $R$.

(ii) The relationen $R$ is not symmetric. Give the smallest set of pairs which must be added to $R$ to make it symmetric.

(iii) The relationen $R$ is not reflexive. Give the smallest set of pairs which must be added to $R$ to make it reflexive.

(iv) The relationen $R$ is not transitive. Give the smallest set of pairs which must be added to $R$ to make it transitive.

(v) Justifying your answer, say whether $R$ is anti-symmetric. [2p]

**Question 5**

For each of the following, decide which options are correct and which are incorrect. Give a short explanation for each of your answers.

(a) The number of 8-bit binary strings containing exactly 3 zeros is

(i) $8 \cdot 7 \cdot 6$
(ii) $8 \cdot 7$
(iii) $2^3$
(iv) $8^3$
(v) $2^5 \cdot 3$ [0.75p]

(b) A function $f : \{0, 1, 2\} \to \{0, 1, 2, 3\}$

(i) is never one-to-one
(ii) is always one-to-one
(iii) is never onto
(iv) is always onto [0.75p]

(c) For all sets $A$ and $B$ where $|A| = 5$ and $|B| = 10$ we have that

(i) $|B - A| = 5$

(ii) $|A \cup B| = 15$

(iii) $|A \cup B| < 15$

(iv) $|A \times B| = 75$ [0.75p]

(d) Let $n$ denote an integer. The proposition

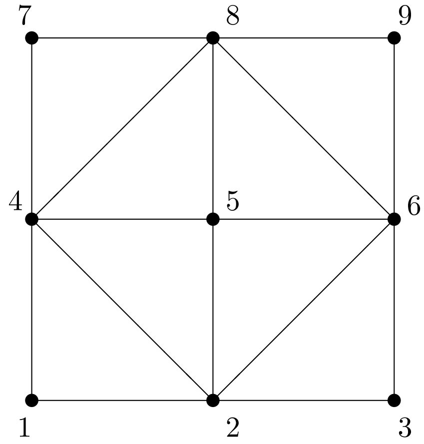
'if $n^2 - 1 < 0$ then $-1 < n < 1$'

is the contrapositive of the proposition

(i) 'if $1 > n > -1$ then $n^2 - 1 \geq 0$'
(ii) '$n^2 - 1 \geq 0$ if $-1 \geq n$ or $n \geq 1$'
(iii) 'if $-1 \geq n$ and $n \geq 1$ then $n^2 - 1 \geq 0$'
(iv) 'if $n^2 - 1 \geq 0$ then $-1 \geq n$ or $n \geq 1$' [0.75p]

**Question 6**

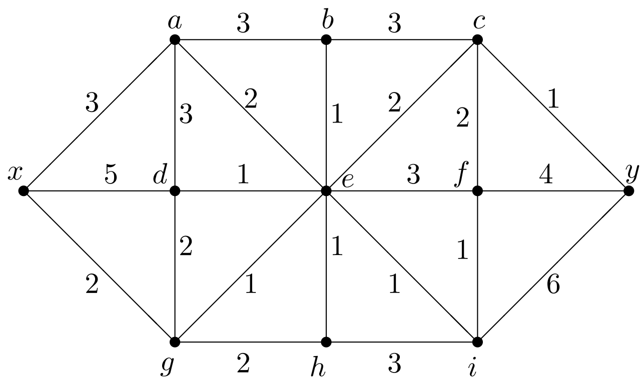
(a) (i) Write down an adjacency matrix for the following graph $G$.




(ii) Explain why the sum of all entries in the adjacency matrix for $G$ is twice the number of edges in $G$.

(iii) Is $G$ is Eulerian? Prove your answer. [1.5p]

(b) Showing all your working, use *Dijkstra's algorithm* to find a shortest path from vertex $x$ to vertex $y$ in the weighted graph below. Take care to show how the vertices become labelled and how the labels change during the run of the algorithm. Finally, give the length of the shortest path. [1.5p]



## Question 7

(a) Let $G$ be a graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and adjacency matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) Draw a picture of $G$.

(ii) Justifying your answer, say whether $G$ is *bipartite*. [1.5]

(b) Define what it means for two simple graphs to be *isomorphic*. [0.5]

(c) Let $H$ be the graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edge set

$$E(H) = \{(1,2),\ (2,3),\ (2,5),\ (3,4),\ (3,6),\ (4,5)\}.$$

Justifying your answer, say whether $G$ and $H$ are isomorphic. [1]

## Question 8

(a) (i) What is meant by saying that $a \equiv b \pmod 3$?

(ii) Prove that if $a \equiv b \pmod 3$ and $c \equiv d \pmod 3$ then $ad \equiv bc \pmod 3$.

(iii) List the 3 congruence classes of the congruence relation $a \equiv b \pmod 3$ on $\mathbb{Z}$ and describe the set $\mathbb{Z}_3$.

(iv) Showing your working, compute the addition and multiplication tables for $\mathbb{Z}_3$. [1.5p]

(b) (i) Use *Euclid's algorithm* to show that $\gcd(1721, 1271) = 1$.

(ii) Find two integers $s$ and $t$ such that

$$1271s + 1721t = 1.$$

(iii) Showing all your working, find all solutions $[x] \in \mathbb{Z}_{1721}$ of the equation

$$[1271] \odot [x] = [3].$$

[1.5p]