MID SWEDEN UNIVERSITY

TFM

Examination 2007

MAAA99 Algebra och Diskret Matematik A (English)

Time: 5 hours

Date: 9 January 2007

There are EIGHT questions on this paper and each question carries three points. The maximum number of points available is 24. The points for each part of a question are indicated at the end of the part in [ ]-brackets. Ten points are needed for the mark G and eighteen points are required for the mark VG.

The candidates are advised that they must always show their working, otherwise they will not be awarded full marks for their answers.

The candidates are further advised to start each of the eight questions on a new *page and to clearly label all their answers.*

This is a closed book examination. No books, notes or mobile telephones are allowed in the examination room. Note that a collection of formulas is attached to the paper.

Electronic calculators may be used provided they cannot handle formulas. The make and model used must be specified on the cover of your script.

GOOD LUCK!!

Make sure that you are doing the right Discrete Mathematics A paper! This paper is for the course which used

[Johnsonbaugh]

as coursebook.

Question 1

(a) Express the decimal number $(1273)_{10}$

(i) in binary;

(ii) as a hexadecimal number. [1p]

(b) Express the following formulas by using Σ-notation.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

[1p]

(c) Use the formulas from (b) to compute the following sum.

$$\sum_{j=3}^{150}((j+1)^3 - 5j + 2).$$

[1p]

Question 2

(a) Let A,B and C be subsets of the universal set $\mathcal{U}$. Shade the area

$$X = (A \cap (\overline{B \cup C})) \cup ((B \cap C) \cap \overline{A})$$

on a Venn diagram. [1p]

(b) Let $Y = \{d \in \mathbb{Z}^+ \ : \ d|20\}$

(i) Give the set Y by listing its elements.

(ii) Let $f : Y \to \mathbb{Z}$ be a function given by $f(x) = x^2$. Make a table to show the function f and use the table to decide whether f is

- one-to-one;
- onto. [1p]

(c) A deck of cards consists of 52 cards with 13 different values of card in each of 4 suits. A Poker hand consists of 5 cards from the deck. A Poker hand is a *full house* if it consists of three cards of one value and a pair of cards of another value. How many ways are there to draw a full house from a deck of cards? [1p]

Question 3

(a) Explain what it means for a relation R on a set X to be

(i) reflexive;

(ii) symmetric;

(iii) transitive;

(iv) an equivalence relation. [1p]

(b) Define the relation R on the set $X = \{-3, -2, -1, 0, 1, 2, 3, 4\}$ by

$$(x_1, x_2) \in R \text{ iff } x_1 \leq x_2.$$

Decide whether R is

(i) a partial order;

(ii) an equivalence relation. [1p]

(c) Give an example of an equivalence relation on $\mathbb{Z}$ which has as its two equivalence classes the odd integers and the even integers. Explain why your relation is an equivalence relation. [1p]

Question 4

Give a proof by induction to show that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all integers $n \geq 1$. [3p]

Question 5

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$a_{n+2} = -6a_{n+1} - 9a_n \quad \text{for } n \geq 0$$

and the two initial terms $a_0 = -1$ and $a_1 = 1$.

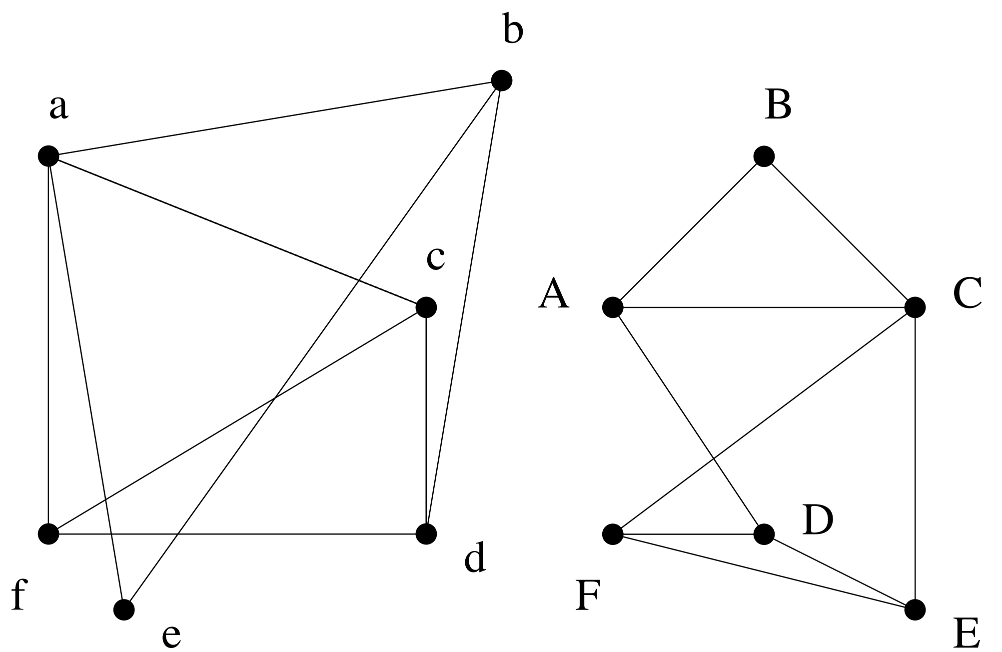
(a) Showing all your working, use the recurrence relation to compute a_2, a_3, a_4, a_5 and a_6. [1p]

(b) Solve the recurrence relation, i.e. give a_n as a function of n for all $n \geq 0$. [2p]

Question 6

(a) (i) Define what it means for two simple graphs to be isomorphic.

(ii) Find an isomorphism between the graphs G_1 and G_2 below. [1p]

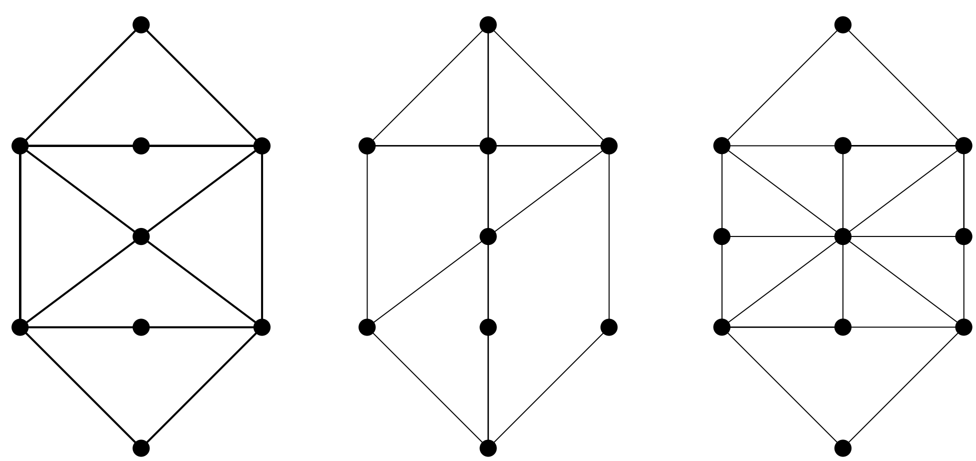


G_1 G_2

(b) Justifying your answers, say for each of the following graphs $H1$, $H2$ and $H3$ whether it is

(i) Hamiltonian;

(ii) Eulerian.

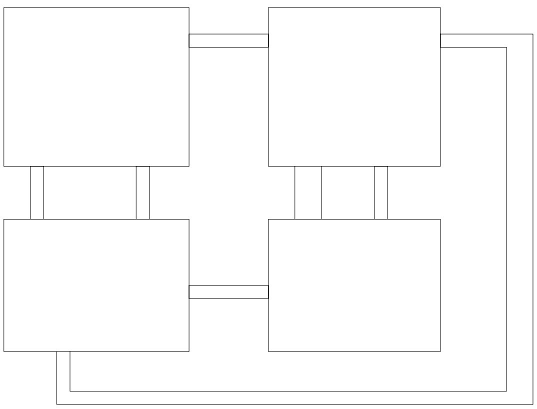


H_1 H_2 H_3

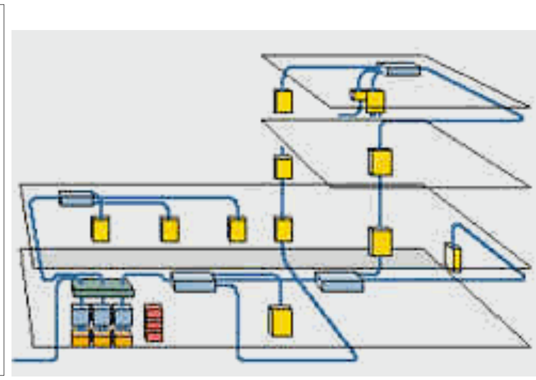
[1p]

(c) Little Sam, age 3 years, has a friend Pia who has built a pneumatic despatch system between the four rooms in her house. Sam has decided that he wants a pneumatic despatch system too as he has seen how much fun Pia's family has by sending messages between rooms via the tubes. Sam has just three rooms in his house, and his ideal system has three tubes in his bedroom and just two tubes in each of the other two rooms. Sam is rather unhappy as he cannot figure out how to design his system. One fine day Sam's friend Pia visits and after a little thought she declares Sam's ideal system to be impossible. Is that true? [1p]

Hint: Sam has made the following sketch of the pneumatic despatch system in Pia's house:



Sam's sketch of Pia's system



Pia's real system

Question 7

(a) Use Euclid's Algorithm to find the greatest common divisor of 390 and 616. [0.5p]

(b) Find a pair of integers s and t such that

$$390s + 616t = \gcd(390, 616).$$

[0.5p]

(c) Find all integer solutions to the congruence

$$616x \equiv 4 \pmod{390}.$$

[0.5p]

(d) Find all integer solutions to the congruence

$$616x \equiv 5 \pmod{390}.$$

[0.5p]

(e) Find all solutions $[x] \in \mathbb{Z}_{390}$ of the equation

$$[616] \odot [x] = [8].$$

[1p]

Question 8

(a) (i) Describe either *Kruskal's Algorithm* or *Prim's Algorithm* for finding a minimum spanning tree in a weighted graph.

(ii) Illustrate your description in (i) by using your chosen algorithm to find a minimum spanning tree in the weighted graph below.

(iii) Give the weight of the minimum spanning tree found. [1.5p]

(b) (i) Describe *Dijkstra's Algorithm* for finding the shortest path from a vertex x to a vertex y in a weighted graph.

(ii) Illustrate your description of the algorithm by using it to find the shortest path from the vertex x to every other vertex in the graph below. [1.5p]