MID SWEDEN UNIVERSITY

TFM

Examination 2007

MAAA99 Algebra och Diskret Matematik A (English)

Time: 5 hours

Date: 23 March 2007

There are EIGHT questions on this paper and each question carries three points. The maximum number of points available is 24. The points for each part of a question are indicated at the end of the part in [ ]-brackets. Ten points are needed for the mark G and eighteen points are required for the mark VG.

The candidates are advised that they must always show their working, otherwise they will not be awarded full marks for their answers.

The candidates are further advised to start each of the eight questions on a new *page and to clearly label all their answers.*

This is a closed book examination. No books, notes or mobile telephones are allowed in the examination room. Note that a collection of formulas is attached to the paper.

Electronic calculators may be used provided they cannot handle formulas. The make and model used must be specified on the cover of your script.

GOOD LUCK!!

Make sure that you are doing the right Discrete Mathematics A paper! This paper is for the course which used

[Johnsonbaugh]

as coursebook.

Question 1

(a) Express the number

$$t = 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8$$

(i) in binary;

(ii) in hexadecimal;

(iii) in base 3. [1.5p]

(b) (i) Write the sum

$$S = 21 + 23 + 25 + 27 + 29 + \ldots + 1001$$

using Σ-notation.

(ii) Calculate the sum S. [1.5p]

Question 2

Let $M = \overline{X \cup Y}$ where
$X = \{ x \in \mathbb{Z}^+ \ : \ x \geq 10 \}$ and $Y = \{ n \in \mathbb{Z}^+ \ : \ n \equiv 1 \pmod 2 \}$.

(a) Express the set M

(i) by the rules of inclusion method;

(ii) by the listing method. [1.5p]

(b) Let $f : M \to \mathbb{Z}$ be the function given by the rule $f(m) = 2m^2$. Make a table to show the function f and use the table to decide whether f is one-to-one and/or onto. [1.5p]

Question 3

(a) Let $\mathcal{U} = \{1, 2, 3, 4, 5, \ldots, 1000\}$,
let $A = \{ n \in \mathcal{U} \ : \ 3|n \}$ and $B = \{ n \in \mathcal{U} \ : \ n \text{ is even } \}$.

(i) Calculate $|A|$, $|B|$ and $|A \cap B|$.

(ii) Find the number of odd, positive integers less than 1000 divisible by 3. [2p]

(b) A deck of cards consists of 52 cards with 13 different values of card in each of 4 suits. A *pair* consists of two cards of the same value. How many pairs are there in a deck of cards? [1p]

Question 4

(a) Use Euclid's algorithm to find the greatest common divisor of 728 and 259. [0.75p]

(b) Find integers s and t such that

$$728s + 259t = \gcd(728, 259).$$

[0.75p]

(c) Find all integer solutions to the following congruences.

(i) $259x \equiv 7 (\operatorname{mod} 728)$;

(ii) $728x \equiv 7 (\operatorname{mod} 259)$;

(iii) $259x \equiv 1 (\operatorname{mod} 728)$. [1p]

(d) Find all solutions $[x] \in \mathbb{Z}_{728}$ of the equation

$$[259] \odot [x] = [21].$$

[0.5p]

Question 5

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$a_{n+1} = a_n + \frac{1}{(2n+1)(2n+3)} \qquad \text{for } n \geq 0,$$

and the initial term $a_0 = 0$.

(a) Showing all your working, use the recurrence relation to compute a_1, a_2, a_3, a_4 and a_5. [1p]

(b) Prove by induction that

$$a_n = \frac{n}{2n+1}$$

for all $n \geq 0$. [2p]

Question 6

(a) Let a, b and c be integers. Prove that

(i) $2|(a+a)$;

(ii) if $2|(a+b)$ then $2|(b+a)$ also;

(iii) if $2|(a+b)$ and $2|(b+c)$ then $2|(a+c)$. [1.5p]

(b) Explain what it means for a relation R on a set S to be an *equivalence relation.* [0.5p]

(c) Let R be a relation on $\mathbb{Z}$ given by

$$(x, y) \in R \text{ iff } 2|(x+y).$$

Prove that R is an equivalence relation and list the equivalence classes of R. [1p]

Question 7

Let G be a graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Draw a picture of G. [0.5p]

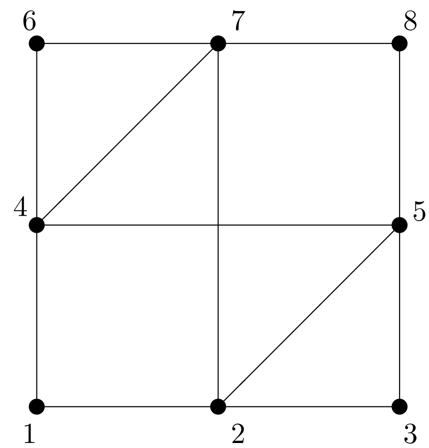
(b) Define what it means for a graph to be a *tree.* [0.5p]

(c) What does it mean for two simple graphs H_1 and H_2 to be isomorphic? [0.5p]

(d) Find three non-isomorphic spanning trees in the graph G. Justify why your trees are non-isomorphic. [1.5p]

Question 8

(a) (i) List the degree sequence of the following graph G.



(ii) Explain why the sum of all the entries in the degree sequence of G is twice the number of edges in G.

(iii) Find an Euler cycle in G. [1.25p]

(b) Define what it means for a graph to be *connected*. [0.25p]

(c) Let H be a Hamiltonian graph, let e be any edge in H and H_e be the graph H with the edge e deleted.

(i) Explain why both H and H_e are connected.

(ii) Hence prove that the graph below cannot be Hamiltonian. [1.5p]