MID SWEDEN UNIVERSITY

TFM

Examination 2007

MAAA99 Algebra and Discrete Mathematics (English)

Time: 5 hours

Date: 7 June 2007

There are EIGHT questions on this paper and each question carries three points. The maximum number of points available is 24. The points for each part of a question are indicated at the end of the part in [ ]-brackets. Ten points are needed for the mark G and eighteen points are required for the mark VG.

The candidates are advised that they must always show their working, otherwise they will not be awarded full marks for their answers.

The candidates are further advised to start each of the eight questions on a new *page and to clearly label all their answers.*

This is a closed book examination. No books, notes or mobile telephones are allowed in the examination room. Note that a collection of formulas is attached to the paper.

Electronic calculators may be used provided they cannot handle formulas. The make and model used must be specified on the cover of your script.

GOOD LUCK!!

Make sure that you are doing the right Discrete Mathematics A paper! This paper is for the course which used

[Johnsonbaugh]

as coursebook.

Question 1

(a) Express the binary number $(110001101010101111)_2$

(i) in hexadecimal;

(ii) in base 10. [1p]

(b) Express the following formula by using Σ-notation.

$$1 + 3 + 5 + 7 + 9 + \ldots + (2n + 1) = (n + 1)^2.$$

[1p]

(c) Use the formula from (b) to compute the sum

$$\sum_{n=100}^{300} (2n - 1).$$

[1p]

Question 2

The sequence $\{S_n(x)\}_{n=1}^{\infty}$ is defined by

$$S_n(x) = \sum_{r=0}^{n} x^r, \qquad \text{for } x \in \mathbb{R}, n \geq 1.$$

(a) Showing all your working, compute $S_1(x), S_2(x), S_3(x)$ and $S_4(x)$. [1p]

(b) Prove by induction that

$$S_n(x) = \frac{x^{n+1} - 1}{x - 1}$$

for all $x \neq 1, n \geq 1$. [1.5p]

(c) Compute $S_n(1)$ for $n \geq 1$. [0.5p]

Question 3

(a) Let A,B and C be subsets of the universal set $\mathcal{U}$.

(i) Shade the area corresponding to the set $X = (A \cap B) \cup \overline{C}$ in a Venn diagram.

(ii) Let $Y = (A \cap \overline{C}) \cup (\overline{C \cup \overline{B}})$. Decide whether $X = Y$ for *all* possible choices of A, B and C. Prove your answer! [1.5p]

(b) Give the set

$$M_1 = \{\tfrac{1}{3}, \tfrac{3}{9}, \tfrac{5}{27}, \tfrac{7}{81}, \tfrac{9}{243}, \ldots\}$$

by using rules of inclusion.

[0.5p]

(c) Give the following sets by the listing method.

(i) $M_2 = \{x \in \mathbb{Z} \mid 5x^2 + 2x - 3 = 0\}$;

(ii) $\mathcal{P}(S)$, where $S = \{x \in \mathbb{Z} \mid x(x-1)(x-2) = 0\}$.

[1p]

Question 4

(a) Explain what it means for a relation R on a set S to be an equivalence relation. [0.5p]

(b) Let R be the relation on the set $S = \{w, x, y, z\}$ given by

$$R = \{(w, w), \quad (x, y), \quad (y, z), \quad (z, x)\}$$

(i) Draw the relation digraph for R.

(ii) The relation R is not reflexive. List the smallest set of pairs which must be be added to R to make it reflexive.

(iii) The relation R is not symmetric. List the smallest set of pairs which must be be added to R to make it symmetric.

(iv) The relation R is not transitive. List the smallest set of pairs which must be be added to R to make it transitive.

(v) Suppose that all the pairs you listed in (iii) and (iv) are added to R, does this make R an equivalence relation? Justify your answer! [2.5p]

Question 5

(a) Give the truth table for $p \Rightarrow q$. [0.5p]

(b) Give the truth table for $\neg q \vee p$. [0.5p]

(c) Let a, b and n be positive integers. For each of the following propositions say whether it is true or false and give a counterexample for each false proposition.

(i) $n > 0$;

(ii) $n > 0 \Rightarrow n \geq 1$;

(iii) $n \geq 1$;

(iv) $n > 1 \Rightarrow a > b$;

(v) $n < 1 \Rightarrow a > b$;

(vi) $a > 1$ or $b > 4 \Rightarrow ab \geq 10$;

(vii) $a > 1$ and $b > 4 \Rightarrow ab \geq 10$;

(viii) $n|ab \Rightarrow n|a$ or $n|b$. [2p]

Question 6

(a) How many positive integers less than 1000 are

(i) a multiple of 2?

(ii) not a multiple of 3?

(iii) not a multiple of 2, 3 or 7? [1.5p]

(b) A PIN-number for a credit card consists of an ordered sequence of 4 digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many PIN-numbers exist

(i) if repeated digits are allowed?

(ii) with four distinct digits?

(iii) which do not contain the digit 9? [1.5p]

Question 7

Let G be the graph with vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and let H be the graph with vertex set $V(H) = \{a, b, c, d, e, f\}$ and edge set

$$E(H) = \{ab, ac, bd, be, cd, ce, fd, fe\}.$$

(a) Draw G and H.

(b) What does it mean for a graph to be *bipartite*?

(c) Justifying your answer, say whether G is bipartite.

(d) When are two simple graphs said to be *isomorfic*?

(e) Justifying your answer, say whether G and H are isomorfic. [3]

Question 8

(a) (i) Define what it means to say that $a \equiv b \pmod 6$.

(ii) Prove that if $a \equiv b \pmod 6$ and $c \equiv d \pmod 6$ then $ad \equiv bc \pmod 6$.

(iii) Give the 6 congruence classes for the congruence relation $a \equiv b \pmod 6$ on $\mathbb{Z}$ and describe the set $\mathbb{Z}_6$.

(iv) Give the addition and multiplication tables for $\mathbb{Z}_6$. [1.5p]

(b) (i) Using *Euclid's Algorithm,* show that $\gcd(1721, 1271) = 1$.

(ii) Find two integers s and t such that

$$1271s + 1721t = 1.$$

(iii) Showing all your working, find all solutions $[x] \in \mathbb{Z}_{1721}$ of the equation

$$[1271] \odot [x] = [2].$$

[1.5p]