MA0146 2 November 2007 - answers.

Question 1

(a)(i) 101010101010

(ii) In decimal $t = 199290$, continuous division by 2 yields the binary representation $t = (110000101001111010)_2$

(iii) $t = 199290$, continuous division by 16 yields the hexadecimal representation $t = (30A7A)_{16}$.

(b) $s = \sum_{n=1}^{500} 2n \cdot 3^{2n-1}$

(c) $\sum_{n=7}^{207} (5n-2) = 5\sum_{n=1}^{207} n - \sum_{n=1}^{207} 2 - 5\sum_{n=1}^{6} n + \sum_{n=1}^{6} 2 = \frac{5 \cdot 207 \cdot 208}{2} - 2 \cdot 207 - \frac{5 \cdot 6 \cdot 7}{2} + 12$

$= 107133.$

Question 2

(a)(i) $\exists x \in \mathbb{Z}$ such that x is a unit or x is composite.

(ii) if $x < 2$ then $x \leq 1$ or $x \in \mathbb{R} \setminus \mathbb{Z}$.

(b) (i) $R = \{0, 8, 32, 128, 512, 2048, \ldots\}$.

(ii) $R \setminus \{0\} = \{8, 32, \ldots\} = \{2^{2x+1} \mid x \in \mathbb{Z}_+\}$.

(iii) f is not onto as e.g. $1 \notin R$.

(iv) f is not 1-1 as e.g. $f(1) = f(3) = 0$.

(v) f is not invertible as it is not 1-1, onto.

Question 3

(a) (i) $a \equiv b \pmod 3$ iff $3 \mid (a-b)$

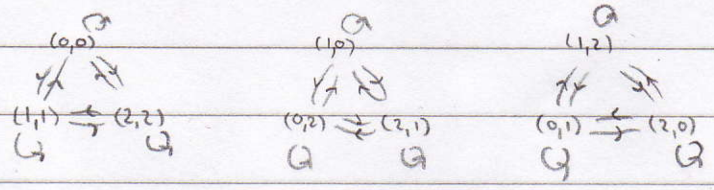
(ii) $a \equiv b \pmod 3$, $x \equiv y \pmod 3$ $\Rightarrow$ $3 \mid (a-b)$, $3 \mid (x-y)$ $\Rightarrow$

$\exists\, k, l \in \mathbb{Z}$ such that $a-b = 3k$ and $x-y = 3l$, hence

$(a+x) - (b+y) = 3k + 3l = 3(k+l)$ and $a+x \equiv b+y \pmod 3$.

(b) (i) $S = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$

(ii)



(iii) Suppose $(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$

then $x_1 - x_2 \equiv y_1 - y_2 \pmod 3$ and $x_2 - x_3 \equiv y_2 - y_3 \pmod 3$

By (a)(ii) we get $(x_1 - x_2) + (x_2 - x_3) \equiv (y_1 - y_2) + (y_2 - y_3) \pmod 3$

yielding $x_1 - x_3 \equiv y_1 - y_3 \pmod 3$ and thus $(x_1, y_1) R (x_3, y_3)$

and R is transitive.

(iv) R is transitive by (iii), R is reflexive as every vertex in the relation digraph has a loop. R is symmetric as the relation digraph contains 'double arcs' only. Hence R is an equivalence relation.

(v) From the relation digraph we get the three equivalence classes:

$\{(0,0), (1,1), (2,2)\}$ $\{(1,0), (2,1), (0,2)\}$ $\{(0,1), (1,2), (2,0)\}$

Question 4

(a) $u_1 = 3u_0 + 3^2 = 3 \cdot 0 + 9 = \underline{\underline{9}}$, $u_2 = 3u_1 + 3^3 = 3 \cdot 9 + 27 = \underline{\underline{54}}$, $u_3 = \underline{\underline{243}}$, $u_4 = \underline{\underline{972}}$, $u_5 = \underline{\underline{3645}}$

(b) <u>BASE</u> $(n=0)$ LHS $= u_0 = 0$, RHS $= 0 \cdot 3^{0+1} = 0$ so LHS = RHS for $n=0$.

<u>IND. HYP.</u> Assume $u_k = k3^{k+1}$ for some $k \geq 0$.

<u>IND. STEP</u> We must prove $u_{k+1} = (k+1)\,3^{k+2}$ $(*)$

But LHS$(*)$ $= u_{k+1} = 3u_k + 3^{k+2}$ by the rec. rel

$= 3 \cdot k \cdot 3^{k+1} + 3^{k+2}$ by the ind. hyp

$= k3^{k+2} + 3^{k+2} = (k+1)\,3^{k+2} =$ RHS$(*)$

So LHS=RHS for $n = k+1$ and thus for all $n \geq 0$ by induction.

Question 5

(a) (i) $9 \cdot 8 \cdot 7 \cdot 6 = 3024$

(ii) $5 \cdot 8 \cdot 7 \cdot 6 = 1680$

(iii) $8 \cdot 7 \cdot 6 \cdot 5 = 1680$

(iv) $5 \cdot 4 \cdot 7 \cdot 6 = 840$

(v) By P.I.E. we get from (ii), (iii) and (iv) : $1680 + 1680 - 840 = \underline{\underline{2520}}$

(vi) From (v) and (iv) we get $2520 - 840 = \underline{\underline{1680}}$

(b) (i) $\underline{\underline{6!}}$

(ii) $4!$ strings contain 654, 3, 2, 1

$4!$ strings contain 543, 6, 2, 1

$5!$ strings contain 21, 3, 4, 5, 6

$3!$ strings contain 6543, 2, 1

$3!$ strings contain 654, 21, 3

$3!$ strings contain 543, 21, 6

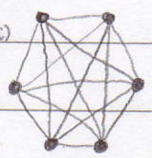
$2!$ strings contain 6543, 21

Hence by PIE we get that $24 + 24 + 120 - 6 - 6 - 6 + 2 = \underline{\underline{152}}$ strings contain at least one of them.

Question 6

(a) A path of length ℓ in a simple graph G is an alternating sequence of $\ell+1$ vertices and ℓ edges $(v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell)$ in which e_i is incident to v_{i-1}, v_i for $i = 1, \ldots, \ell$.

(b)(i)



(b)(ii) K_6 has $\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ edges.

(b)(iii) v has $6-2$ neighbours other than w, and a path of length 2 must go via one of these. Hence $\underline{\underline{4}}$ paths of length 2 from v to w.

(b)(iv) $\underline{\underline{n-2}}$ by a similar argument to (iii)

(c) 

G_1

and 

G_2

are not isomorphic as G_2 has two adjacent vertices of degree 3 while G_1 has not.

(d) Not possible, a tree on n vertices has $n-1$ edges, a graph with degree sequence 4, 3, 3, 2, 1, 1 would have 6 vertices and 7 edges.

Question 7

(a) (i) See book or lecture notes for a description of the algorithms.

(ii)-(iii) Solution depends on algorithm chosen, but all MSTs have weight $\underline{\underline{19}}$.

Kruskal e.g. chooses xa, xg, be, ec, fk, eh, ab, de, hi, cf.

(b) (i) See book or lecture notes for description of Dijkstra's algorithm.

(ii) The vertices get permanent labels in one of the following 4 orders:

x, a/g, g/a, b, e, h/c, c/h, d, i, f, k. The lengths of the shortest path to a vertex is given by its permanent label, the path is found by tracing it using the permanent labels:

x-a length 1,

x-g length 1

x-a-b length 3

x-a-b-e length 4

x-g-h length 5

x-a-b-e-c length 5

x-a-b-e-d length 6

x-g-h-i length 9

x-a-b-e-f length 11

x-a-b-e-c-k length 12

Question 8

(a) Using Euclid's algorithm forwards and backwards yields $\gcd(2007, 1857) = 3$

and $\underline{3} = 260 \cdot \underline{2007} + (-281) \cdot \underline{1857}$

(b) by (a) $[2 \cdot (-281)]_{2007} = [1445]_{2007}$ is a solution, the other two are thus

$[1445 + \frac{2007}{3}]_{2007} = [107]_{2007}$ and $[107 + \frac{2007}{3}]_{2007} = [776]_{2007}$

(c) by (b) the solutions are all $x \in [107]_{669} = \{x \in \mathbb{Z} \mid x = 107 + k \cdot 669,\ k \in \mathbb{Z}\}$.

(d) By (a) $\underline{1} = 260 \cdot \underline{669} + (-281) \cdot \underline{619}$, so one congruence class in $\mathbb{Z}_{669}$ solves this congruence, namely $[5 \cdot (-281)]_{669} = [602]_{669}$. The solutions are thus

$x \in [602]_{669} = \{x \in \mathbb{Z} \mid x = 602 + k669,\ k \in \mathbb{Z}\}$.

Question 9

(a) By formula collection $\sum_{r=0}^{n} 2^r = \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1$

so $\sum_{r=1}^{n} 2^r = 2^{n+1}-1-1 = 2^{n+1}-2$

(b) BASE $n=1$

LHS $= \sum_{r=1}^{1} 2^{2r-1} = 2$, RHS $= \sum_{r=1}^{2} 2^r - \sum_{r=1}^{1} 2^{2r} = 2+4-4 = 2$ so LHS=RHS for $n=1$.

IND. HYP. Assume $\sum_{r=1}^{k} 2^{2r-1} = \sum_{r=1}^{2k} 2^r - \sum_{r=1}^{k} 2^{2r}$ for some $k \geq 1$.

IND. STEP. We must prove $\sum_{r=1}^{k+1} 2^{2r-1} = \sum_{r=1}^{2k+2} 2^r - \sum_{r=1}^{k+1} 2^{2r}$ ⊛

But RHS⊛ $= \sum_{r=1}^{2k+2} 2^r - \sum_{r=1}^{k+1} 2^{2r} = 2^{2k+1} + \cancel{2^{2k+2}} + \sum_{r=1}^{2k} 2^r - \sum_{r=1}^{k} 2^{2r} - \cancel{2^{2k+2}}$

$= 2^{2k+1} + \sum_{r=1}^{k} 2^{2r-1}$ by the ind. hyp.

$= \sum_{r=1}^{k+1} 2^{2r-1} =$ LHS⊛

So RHS=LHS for $n=k+1$ and thus for all $n \geq 1$ by induction.

(c) By (b) $\sum_{r=1}^{n} 2^{2r-1} = \sum_{r=1}^{2n} 2^r - \sum_{r=1}^{n} 2^{2r} = \sum_{r=1}^{2n} 2^r - 2\sum_{r=1}^{n} 2^{2r-1}$, hence

$3\sum_{r=1}^{n} 2^{2r-1} = \sum_{r=1}^{2n} 2^r = 2^{2n+1}-2$ by (a)

and from this follows that $\sum_{r=1}^{n} 2^{2r-1} = \frac{2^{2n+1}-2}{3}$.