MA055G & MA056G
Introduktionskurs i matematik

Lecture Notes 7

Pia Heidtmann
080916

The complex numbers

- Recall
$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$$

  On $\mathbb{Z}$ we can do the arithmetic operations $+, -$ and $\times$.

- The rational numbers $\mathbb{Q}$ we 'created' to be able to do division as well:
$$\mathbb{Q} = \{ m/n \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \}.$$

- The real numbers $\mathbb{R}$ were 'created' to fill in the 'gaps' between the rationals on the number line.

- We now define a new number system, which will extend the real numbers, we shall call it the complex numbers and it will be denoted $\mathbb{C}$.

  The reason for needing this extended number system is found in algebra, more specifically in the theory for solving equations:

We already know that if a quadratic polynomial

$$p(x) = x^2 + bx + c$$

has two roots α and β, then

$$p(x) = x^2 + bx + c = (x-\alpha)(x-\beta),$$

and if a cubic polynomial $g(x) = x^3 + ax^2 + bx + c$ has three roots α, β and γ, then

$$g(x) = x^3 + ax^2 + bx + c = (x-\alpha)(x-\beta)(x-\gamma).$$

We say that *these polynomials split into linear factors.*

Unfortunately not all polynomials split nicely into linear factors. For example

$$f(x) = x^2 + 1$$

has no roots, and

$$g(x) = x^3 - x^2 + x - 1 = (x^2+1)(x-1)$$

has just one real root.

<u>AIM</u> We are looking for a number system which has the property that every polynomial of degree n has n roots and thus splits into n linear factors (by the Factor Theorem). This system will be $\mathbb{C}$.

We start by formally defining the complex numbers.

DEFINITION

The complex numbers $\mathbb{C}$ is the set of all ordered pairs of real numbers. That is

$$\mathbb{C} = \{ (a,b) : a,b \in \mathbb{R} \}.$$

Arithmetic in $\mathbb{C}$.

We can define an addition $+$ and a multiplication $\cdot$ on $\mathbb{C}$.

Given any two complex numbers $z_1 = (a,b)$ and $z_2 = (c,d)$,

we define

$$z_1 + z_2 := (a,b) + (c,d) = (a+c, b+d)$$

$$z_1 z_2 := (a,b) \cdot (c,d) = (ac-bd, ad+bc)$$

note that $a+c$, $b+d$, $ac-bd$, $ad+bc$ are real numbers, so z_1+z_2 and $z_1 z_2$ are complex numbers.

We can now prove that $\mathbb{C}$ satisfies all the same rules of arithmetic as do $\mathbb{R}$ and $\mathbb{Q}$ (Note that we call such a structure a field):

kropp

Theorem (Rules of Arithmetic for $\mathbb{C}$)

$\mathbb{C}$ together with its addition $+$ and multiplication $\cdot$ defined above is a field, that is, it satisfies the following rules of arithmetic:

1. Closure Law

For all $z_1, z_2 \in \mathbb{C}$, $z_1 + z_2 \in \mathbb{C}$ and $z_1 \cdot z_2 \in \mathbb{C}$.

2. Identity Elements exist.

- There is a (unique) number $(0,0) \in \mathbb{C}$ such that for all $z \in \mathbb{C}$,
$$z + (0,0) = (0,0) + z = z.$$
- There is a (unique) number $(1,0) \in \mathbb{C}$ such that for all $z \in \mathbb{C}$,
$$z \cdot (1,0) = (1,0) \cdot z = z.$$

3. Additive and multiplicative inverses exist. For each $z \in \mathbb{C}$, there is a (unique) additive inverse $(-z) \in \mathbb{C}$ such that
$$z + (-z) = (-z) + z = (0,0).$$

For each $z \in \mathbb{C}$, $z \neq (0,0)$, there is a unique multiplicative inverse $z^{-1} \in \mathbb{C}$ such that
$$z \cdot z^{-1} = z^{-1} \cdot z = (1,0).$$

4. Associative Laws For all $z_1, z_2, z_3 \in \mathbb{C}$,
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad \text{and} \quad z_1(z_2 z_3) = (z_1 z_2) z_3.$$

5. Commutative Laws For all $z_1, z_2 \in \mathbb{C}$
$$z_1 + z_2 = z_2 + z_1 \quad \text{and} \quad z_1 \cdot z_2 = z_2 \cdot z_1.$$

6. Distributive Laws For all $z_1, z_2, z_3 \in \mathbb{C}$
$$z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Proof Just calculate using the definition and the properties of the real numbers.

For example...

<u>Proof that additive and multiplicative inverses exist:</u>

Let $z = (a,b) \in \mathbb{Z}$

then define $(-z) = (-a,-b) \in \mathbb{C}$.

We have

$$z + (-z) = (a,b) + (-a,-b) = (a-a, b-b) = (0,0)$$

$$(-z) + z = (-a,-b) + (a,b) = (-a+a, -b+b) = (0,0).$$

For $z = (a,b) \in \mathbb{C}$, where $z \neq (0,0)$, define $z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$,

then $$z \cdot z^{-1} = (a,b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(\frac{a^2}{a^2+b^2} - \frac{-b^2}{a^2+b^2}, \frac{-ab}{a^2+b^2} + \frac{ba}{a^2+b^2}\right)$$

$$= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}\right) = (1,0)$$

Similar calculations give

$$z^{-1} \cdot z = (1,0).$$

$\mathbb{C}$ contains $\mathbb{R}$ as a subset, in fact we can identify

$\mathbb{R}$ and the subset $\{(a,0) : (a,0) \in \mathbb{C}\}$

Because $\mathbb{R}$ can be identified with the complex numbers of the form $(a,0)$, we abuse our notation slightly and write

$$(a,0) \text{ as } a.$$

We then <u>define</u>

$$i := (0,1)$$

and we have that

$$(a,b) = a + ib$$

For the complex number $z = (a,b) = a + ib$ we call a the real part of z and b the imaginary part of z.

In symbols we write

$$a = Re(z) \quad \underline{\text{and}} \quad b = Im(z).$$

<u>Note</u> that $b = Im(z)$, though known as the imaginary part, is a <u>real</u> number.

The real numbers are just the complex numbers $z \in \mathbb{C}$ for which $Im(z) = 0$.

Let us work with the number i.

First note

$$i^2 = (0,1)\cdot(0,1) = (0-1, 0+0) = (-1,0)$$

that is $\underline{i^2 = -1}$

If you remember this formula you need not memorize how to multiply complex numbers, for

$$(a,b)\cdot(c,d) = (a+ib)(c+id)$$

$$= ac + iad + ibc + i^2 bd$$

$$= (ac-bd) + i(ad+bc)$$

$$= (ac-bd, ad+bc).$$

For any complex number $z = (a,b) = a+ib$, we define $\bar{z}$, the complex conjugate of z by

$$\bar{z} := (a,-b) = a - ib.$$

We note that

$$z\cdot\bar{z} = (a+ib)(a-ib) = a^2 - i^2 b^2 = a^2+b^2 = (a^2+b^2, 0),$$

that is $z\cdot\bar{z}$ is a <u>real</u> number.

This is a very important fact as it allows us to compute multiplicative inverses of complex numbers without memorizing the formula we gave above...

Let $0 \neq a+ib$, then

$$(a+ib)^{-1} = \frac{1}{(a+ib)} = \frac{a-ib}{(a-ib)(a+ib)} = \frac{a-ib}{a^2+b^2} = \underbrace{\frac{a}{a^2+b^2}}_{\in \mathbb{R}} + i \underbrace{\frac{-b}{a^2+b^2}}_{\in \mathbb{R}}$$

"small" theorem.

↓

Lemma

(a) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Proof (b) We leave (a) as an exercise!

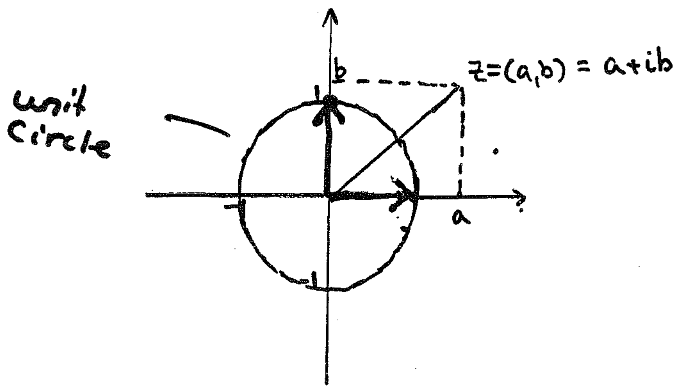
Let $z_1 = a+ib$, $z_2 = c+id$, then

$$\text{RHS} = \overline{a+ib} \cdot \overline{c+id} = (a-ib)(c-id) = (ac-bd) - i(ad+bc)$$

$$= \overline{(ac-bd) + i(ad+bc)} = \overline{(a+ib)(c+id)} = \text{LHS}. \blacksquare$$

Argand diagrams

$z = (a,b) = a + ib$ can be represented graphically in the coordinate system as the point with Cartesian coordinates (a,b). This representation is known as an Argand diagram for z.



Since the x-coordinate of z is $a = Re(z)$, the x-axis is known as the real axis.

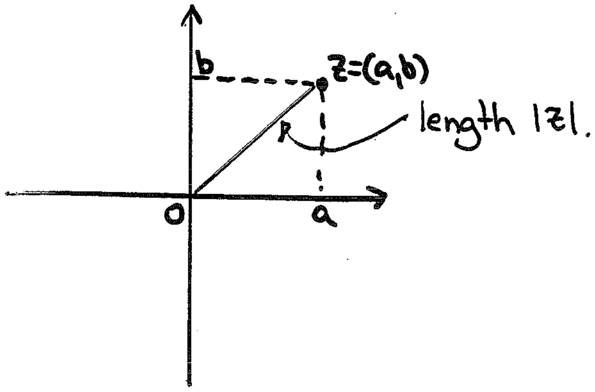
And similarly, because the y-coordinate $b = Im(z)$, the y-axis is known as the imaginary axis.

(absolut) beloppet

We define the norm (also called the absolute value of $z=(a,b) \in \mathbb{C}$ as

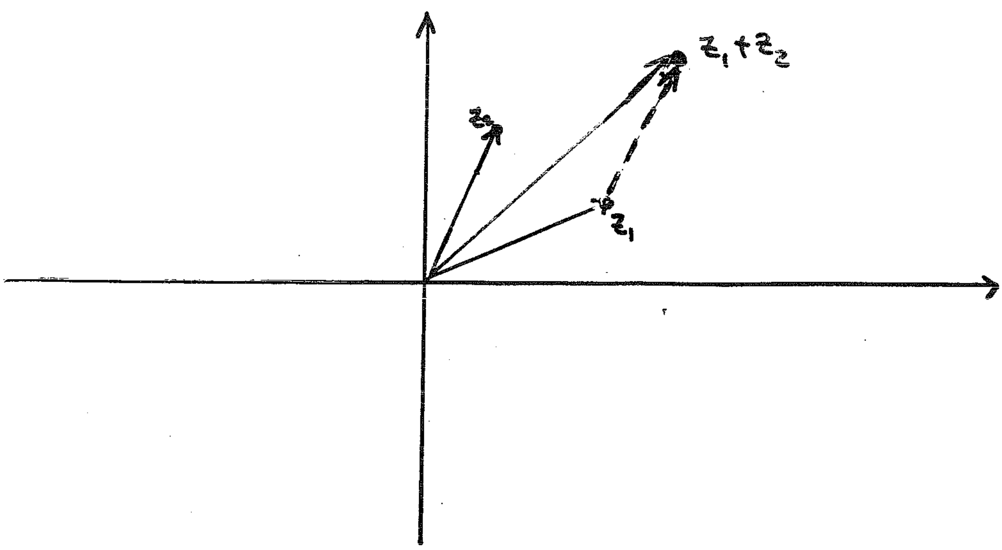
$$|z| = \sqrt{a^2+b^2}$$

Graphically, $|z|$ is the length of the direction vector for z in the Argand diagram, that is the length $\|\vec{Oz}\|$ of the line segment Oz:



The sum of two complex numbers z_1 and z_2 is just the point whose direction vector is the sum of the two direction vectors for z_1 and z_2. Hence by the <u>triangle inequality</u> we have that

$$|z_1+z_2| \leq |z_1|+|z_2|$$



We also note that

LEMMA $|z|^2 = z\bar{z}$

PROOF

$$z\bar{z} = (a+ib)(a-ib) = a^2+b^2 = \left(\sqrt{a^2+b^2}\right)^2 = |z|^2 \qquad \blacksquare$$

COROLLARY $|z_1 z_2| = |z_1||z_2|$

PROOF

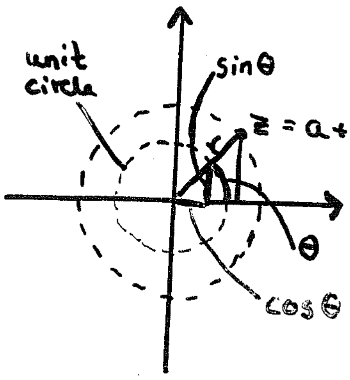
$$|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\bar{z}_1 \cdot \bar{z}_2) = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2$$

and since $|z|$ is a non-negative real number, we thus have

$$|z_1||z_2| = |z_1 z_2| \qquad \blacksquare$$

Polar form of complex numbers (sv: polär form)

Suppose that $z = (a,b) = a+ib \in \mathbb{C}$ has a direction vector of length $r = |z|$ and forms an angle θ with the x-axis:



$$z = a+ib = (a,b) = (r\cos\theta, r\sin\theta) = \underline{\underline{r(\cos\theta, \sin\theta)}}$$

The 'new' form $z = r(\cos\theta, \sin\theta) = \underline{r\cos\theta + i(r\sin\theta)}$ is known as the polar form of z.

The 'old' forms

$$\underline{z = (a,b)} \quad \text{and} \quad \underline{z = a+ib}$$

are known as the Cartesian form of z.

The angle θ is known as the argument of z, and we write

$$\theta = \arg(z)$$

Note that the argument of z is not unique: Adding 2π to θ yields another angle which is also an argument of z.

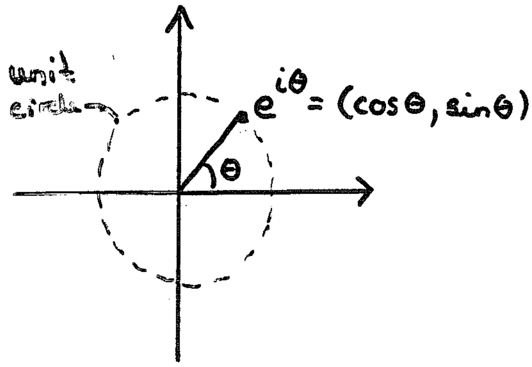
The principal value among the values of $\arg(z)$ is the value in the interval $\underline{]-\pi, \pi]}$ and is denoted by $\mathrm{Arg}(z)$.

If we define the complex number $e^{i\theta}$ by

$$e^{i\theta} := \cos\theta + i\sin\theta,$$

that is

$e^{i\theta}$ is the complex number on the unit circle in the Argand diagram with argument θ



Then, if $z = a+ib = r(\cos\theta, \sin\theta) = r\cos\theta + i(r\sin\theta)$
we can write
$$z = re^{i\theta}$$

Further, we can define the complex exponential function e^z for any $z = a+ib$ as

$$\underline{e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos a + i\sin b)}$$

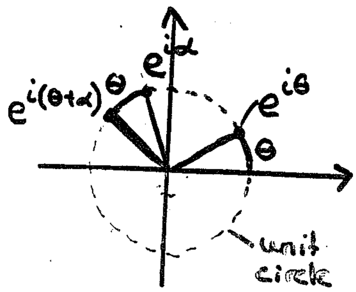
LEMMA

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and

$$w = s(\cos\alpha + i\sin\alpha) = se^{i\alpha}.$$

Then

$$zw = (rs)e^{i(\theta+\alpha)}$$



PROOF

$$zw = (r\cos\theta + ir\sin\theta)(s\cos\alpha + is\sin\alpha)$$

$$= rs\cos\theta\cos\alpha - rs\sin\theta\sin\alpha + i(rs\cos\theta\sin\alpha + rs\sin\theta\cos\alpha)$$

$$= rs\big((\cos\theta\cos\alpha - \sin\theta\sin\alpha) + i(\cos\theta\sin\alpha + \sin\theta\cos\alpha)\big)$$

$$= rs(\cos(\theta+\alpha) + i\sin(\theta+\alpha))$$

$$= rse^{i(\theta+\alpha)} \quad \blacksquare$$

Using this result n times proves ...

De Moivre's Theorem

If n is a positive integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Corollary

If $z = r(\cos\theta, \sin\theta) = r\cos\theta + ir\sin\theta$, then

$$z^n = r^n(\cos(n\theta), \sin(n\theta)).$$

Equations

A (complex) polynomial of degree n is an expression

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

where $a_n \neq 0$ and all coefficients $a_i \in \mathbb{C}$.

A root (nollställe) of p is a (complex) number z_0 such that

$$p(z_0) = 0.$$

First observe: we can divide complex polynomials in the same way as we divide real polynomials because the rules of arithmetic in $\mathbb{C}$ are the same as in $\mathbb{R}$.

We also have

THE FACTOR THEOREM

z_0 is a root of the complex polynomial p if and only if there exists a polynomial g such that

$$p = (z - z_0) g$$

Definition

If the polynomial is divisible by $(z-z_0)^m$, but not by $(z-z_0)^{m+1}$ then z_0 is called a root of multiplicity m.

The Fundamental Theorem of Algebra

Every (complex) polynomial of degree n has precisely n roots if these are counted with multiplicity.

So quadratic equations when solved in $\mathbb{C}$ always have two roots (if we count with multiplicity)

Finding the roots is usually straight forward using the method of <u>completing the square</u>...

Examples

Quadratic Equations

Solve

(a) $z^2+2z+3=0$

(b) $z^2=i$

(c) $z^2=8+6i$

(d) $z^2+4z+2iz+14+64i=0$

Solⁿ (d):

$$z^2 + (4+2i)z + (14+64i) = 0$$

complete the square:

$$(z+(2+i))^2 - (2+i)^2 + 14+64i = 0$$

$$\Updownarrow$$

$$(z+(2+i))^2 = -11-60i \quad (*)$$

Let $w = z+(2+i)$, then $(*)$ becomes

$$w^2 = -11-60i \quad (**)$$

which can be solved by putting

$w = u+iv$ where $u, v \in \mathbb{R}$

then by comparing real and imaginary parts in $(**)$ we get

$$\begin{bmatrix} u^2-v^2 = -11 \\ 2uv = -60 \end{bmatrix}$$

which can be solved to give

$w = 5-i6$ and $w = -5+i6$

$z = w-(2+i) = \underline{\underline{3-7i}}$ or $\underline{\underline{-7+i5}}$

$\underset{\sim}{\text{Sol}^n}$ (a)-(c) (a) $z^2+2z+3=0$

$\Updownarrow$

$(z+1)^2-1+3=0$

$\Updownarrow$

$(z+1)^2=-2$

$\Updownarrow$

$(z+1)^2=2i^2$

$\Updownarrow$

$z+1=\pm\sqrt{2}\,i$

$\Updownarrow$

$\underline{\underline{z=-1\pm\sqrt{2}\,i}}$

(b) $z^2=i$

(solve either by the method of n'th roots) OR

put $z=u+iv$ where u,v are real.

Then

$i=z^2=(u^2-v^2)+i(2uv)$

By comparing real and imaginary parts separately we get

$$\begin{bmatrix} u^2-v^2=0 \\ 2uv=1 \end{bmatrix}$$

$\Downarrow$

$(u+v)(u-v)=0$

$\Downarrow$

$u=v$ or $u=-v$

$\Downarrow$

$2u^2=1$	or $-2u^2=1$
$\Downarrow$	no sol^s
$u=\pm\frac{1}{\sqrt{2}}$	as u is
$\Downarrow$	real
$v=u=\pm\frac{1}{\sqrt{2}}$	

$\underline{\underline{\text{So } z=\pm\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}}$

(c) $z^2=8+6i$. Proceeding as we did in (b) we get $z=u+iv$ and

$$\begin{bmatrix} u^2-v^2=8 \\ 2uv=6 \end{bmatrix}$$

$\Downarrow$

$u=\frac{3}{v}$ ⊗

$\Downarrow$

$v^4+8v^2-9=0$

$\Downarrow$

$v^2=-4\pm5$

$\Downarrow$

$v^2=1$ as v is real

$\Downarrow$

$v=\pm1$

$\Downarrow$

$u=\pm3$ by inserting in ⊗

so $\underline{\underline{z=3+i}}$ or $\underline{\underline{z=-3-i}}$

ROOTS

Find all complex numbers ρ such that

$$\rho^n = z$$

where z is any complex number.

ρ is known as an n'th root of z

Let us first consider the special case $z=1$, that is find all complex numbers ρ such that

$$\rho^n = 1.$$

These are known as the complex n'th roots of unity:

So... Solve $\rho^n = 1$ ⊛

If we let $\rho = re^{i\theta}$, we first observe that $r=1$ (why?)
So complex roots of unity lie on the unit circle in the Argand diagram.

$$\rho = e^{i\theta} = \cos\theta + i\sin\theta$$

Next observe that $1 = 1 + i\cdot 0 = \cos 0 + i\sin 0$

So ⊛ amounts to solving

$$(\cos\theta + i\sin\theta)^n = \cos 0 + i\sin 0,$$

which by De Moivre's Theorem becomes

$$\cos(n\theta) + i\sin(n\theta) = \cos 0 + i\sin 0.$$

So by comparing real and imaginary parts separately we get

$$1 = \cos 0 = \cos(n\theta) \quad \text{and} \quad 0 = \sin 0 = \sin(n\theta).$$

That is,

$$n\theta = k(2\pi) \quad \text{and thus} \quad \underline{\underline{\theta = k\cdot\frac{2\pi}{n}}} \quad k = \ldots -1, 0, 1, 2, 3, \ldots$$

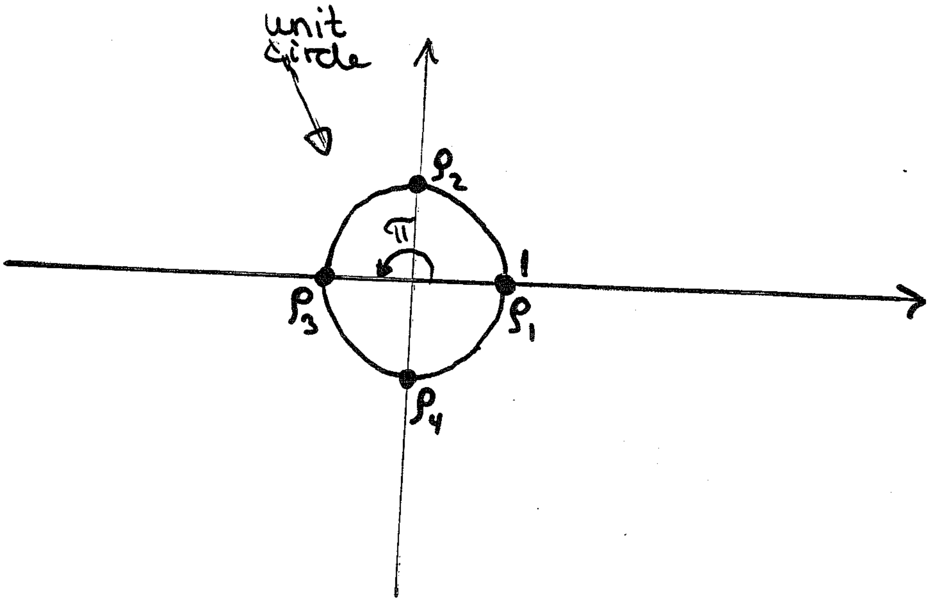
So the n'th roots of unity are $\rho = e^{i\theta}$

where $\theta = k\cdot\frac{2\pi}{n}$, $k = 0, 1, 2, \ldots, n-1, \cancel{n, n+1, n+2, n+3, \ldots}$

These are $\rho = 1$ and $n-1$ other points evenly spaced on the unit circle in the Argand diagram.

Example

Find all ρ such that $\rho^4 = 1$



$\rho_1 = 1$

$\rho_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\rho_3 = \cos\pi + i\sin\pi = -1$

$\rho_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

LEMMA Given any complex number $re^{i\theta}$.

If $z \neq 0$ and $z^n = re^{i\theta}$ then

$$z = \sqrt[n]{r}\, e^{iu}$$

where $u = \frac{\theta}{n} + k\left(\frac{2\pi}{n}\right)$, $k = 0, 1, \dots, n-1$

that is, the n'th roots of $re^{i\theta}$ all lie on the circle in the Argand diagram of radius $\sqrt[n]{r}$ and centre at the origin.

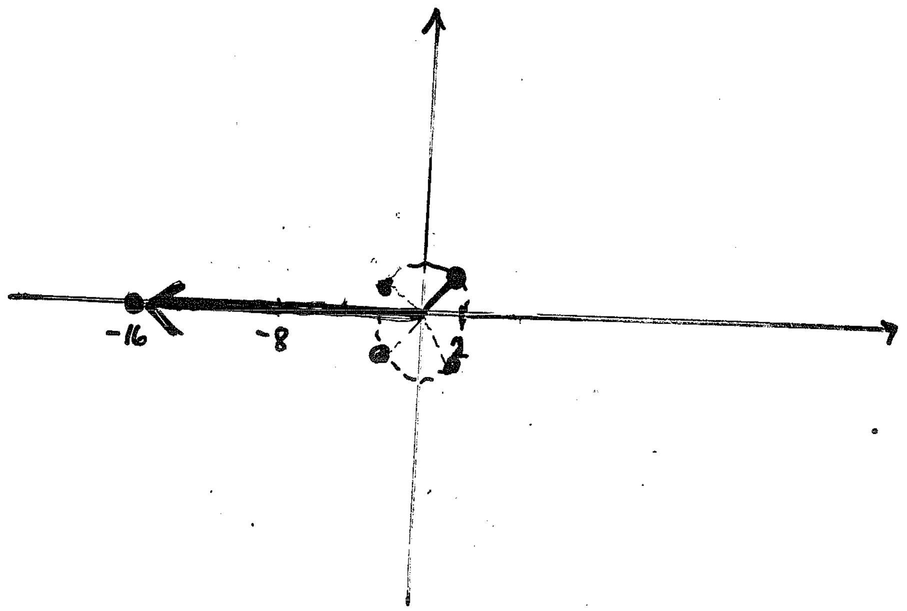
One of the roots has argument $\frac{\theta}{n}$,

there are n roots altogether and they are evenly spaced on the circle.

EXAMPLE

Find all $z \in \mathbb{C}$ such that $z^4 = -16$.

First we need to write -16 in polar form. We do this by locating it on the Argand diagram:



so $\arg(-16) = \underline{\underline{\pi}}$

$|-16| = \underline{\underline{16}}$

Hence $\underline{\underline{-16}} = 16(\cos\pi + i\sin\pi)$

$= \underline{\underline{16e^{i\pi}}}$

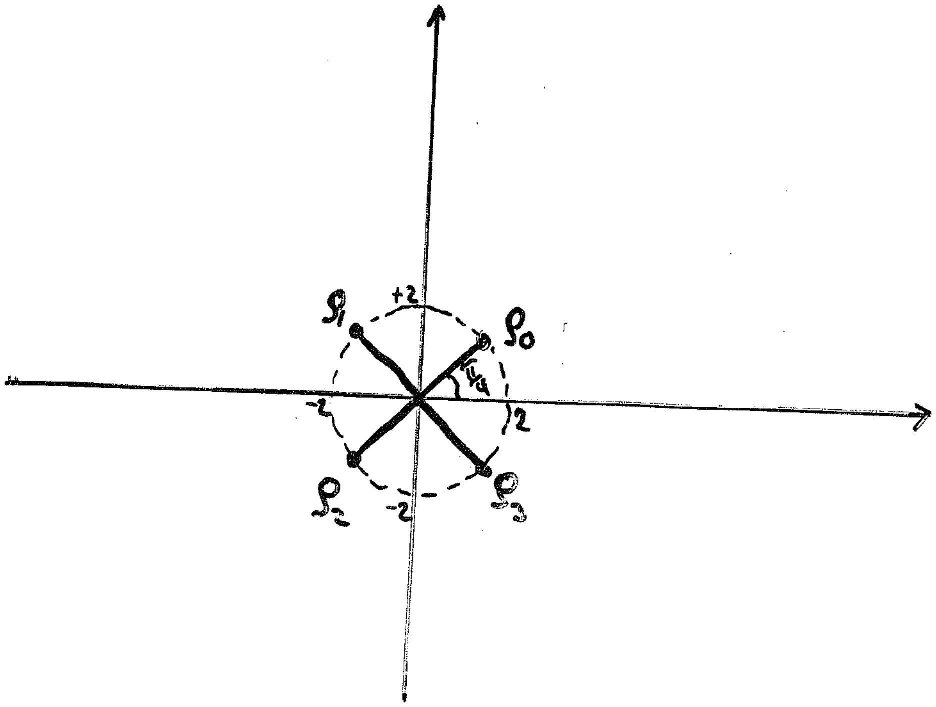
The 'first' fourth root of -16 is thus

$$\rho_0 = 2e^{i\frac{\pi}{4}}$$

and the three others are

$\rho_1 = 2e^{i\left(\frac{3\pi}{4}\right)}$ $\quad\rho_2 = 2e^{i\left(\frac{5\pi}{4}\right)}$ $\quad\rho_3 = 2e^{i\left(\frac{7\pi}{4}\right)}$

On the Argand diagram they are:



$$\rho_0 = 2\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (\sqrt{2}, \sqrt{2}) = \underline{\sqrt{2}(1+i)}$$

$$\rho_1 = 2\left(\cos\frac{3\pi}{4}, \sin\frac{3\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (-\sqrt{2}, \sqrt{2}) = \underline{\sqrt{2}(-1+i)}$$

$$\rho_2 = 2\left(\cos\frac{5\pi}{4}, \sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (-\sqrt{2}, -\sqrt{2}) = \underline{\sqrt{2}(-1-i)}$$

$$\rho_3 = 2\left(\cos\frac{7\pi}{4}, \sin\frac{7\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (\sqrt{2}, -\sqrt{2}) = \underline{\sqrt{2}(1-i)}$$