MA055G & MA056G
Introduktionskurs i matematik

Lecture Notes 8

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SEQUENCES

A sequence is an ordered list of numbers.

More formally a sequence is a function $s : \mathbb{Z}_+ \to \mathbb{R}$.

Example

$$1, 4, 7, 10, 13, 16, 19, 22, 25, \ldots$$

can be described as $s(1)=1$, $s(2)=4$, $s(3)=7, \ldots$
though we usually write $s_1=1$, $s_2=4$, $s_3=7$, ... instead.

The first term of the sequence is known as the initial term.
And we say that the general term, the n'th term, is s_n.

In the example above, the initial term is $s_1=1$

and the general term $s_n = 3n-2$.

In order to define a sequence you can

EITHER give the general term s_n where $n \in \mathbb{Z}_+$

OR give the <u>initial term</u> and a <u>recurrence relation</u> expressing s_n as a function of $s_{n-1}, s_{n-2}, \ldots, s_1$

Example

Consider again the sequence

$$s_1 = 1, \ s_2 = 4, \ s_3 = 7, \ s_4 = 10, \ s_5 = 13, \ \ldots$$

from above.

We could describe this sequence by simply stating that

$$\underline{\underline{s_n = 3n - 2 \quad \text{for all } n \in \mathbb{Z}_+}}$$

thus providing a formula that gives the n'th term.

However, we could also have noticed that each term, after the first one, could be defined as the previous term plus 3. We could therefore define the sequence by giving

a) the initial term $\underline{\underline{s_1 = 1}}$ AND

b) the recurrence relation $\underline{\underline{s_{n+1} = s_n + 3}}$ for $n \in \mathbb{Z}_+$.

Example Some definitions by recurrence relation:

$1, 2, 4, 7, 11, \ldots \qquad u_1 = 1, \quad u_{n+1} = u_n + n, \quad n \in \mathbb{Z}_+$

$3, 4, 6, 9, 13, \ldots \qquad u_1 = 3, \quad u_{n+1} = u_n + n, \quad n \in \mathbb{Z}_+$

Note that the above two sequences have the same recurrence relation, $u_{n+1} = u_n + n$, but are very different due to their different initial terms.

Example

$1, 2, 4, 8, 16, \ldots \qquad u_1 = 1, \quad u_{n+1} = 2u_n, \quad n \in \mathbb{Z}_+$

Example

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

$u_1 = 0, \; u_2 = 1 \qquad u_{n+2} = u_{n+1} + u_n, \quad n \in \mathbb{Z}_+$

Note that the last recurrence relation requires <u>two</u> initial terms to be specified in order to fully define the sequence.

Note that it is not sufficient to list the first few terms in order to define a sequence!

Example

$1, 2, 4, 7, 11, \ldots$

$1, 2, 4, 8, 16, \ldots$

These two sequences have the same first three terms, but are not the same.

Arithmetic Sequences

If each term in a sequence is given by adding some fixed value to the previous term, the sequence is called an arithmetic sequence.

Arithmetic sequences are thus defined by their first term and a recurrence relation of the form

$$u_{n+1} = u_n + d$$

for some constant d, which is known as the (Swedish: differens) common difference of the sequence.

The arithmetic sequence with initial term a and common difference d has the general term

$$u_n = a + (n-1)d, \quad \text{where } n \in \mathbb{Z}_+$$

Example

The arithmetic sequence with initial term 1 and $d = 2$:

$$1, 3, 5, 7, 9, 11, \ldots$$

$u_1 = 1$, $u_{n+1} = u_n + 2$ for $n \in \mathbb{Z}_+$.

The 50th term of this sequence is $1 + (50-1) \times 2 = \underline{\underline{99}}$.

Geometric Sequences

If each term in a sequence is given by multiplying the previous term by some fixed value, the sequence is called a geometric sequence.

Geometric sequences are thus defined by their first term and a recurrence relation of the form

$$u_{n+1} = r u_n$$

for some constant r, which is known as the common ratio (Swedish: kvot) of the sequence.

The geometric sequence with initial term a and common ratio r has the general term

$$u_n = a r^{n-1}, \quad n \in \mathbb{Z}_+.$$

Example

The geometric sequence with initial term 1 and $r = 3$:

$$1, 3, 9, 27, 81, 243, \ldots$$

$u_1 = 1$, $u_{n+1} = 3u_n$ for $n \in \mathbb{Z}_+$.

The 50th term of this sequence is $1 \cdot 3^{50-1} = \underline{\underline{3^{49}}}$

Arithmetic Sums

The *arithmetic sum* with initial term a, common difference d and n terms is the sum of n terms of an arithmetic sequence, that is

$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d).$$

This arithmetic sum is

$$\frac{n}{2} \times (\text{first term } + \text{last term }),$$

which is equal to

$$\frac{n}{2} \times (2a + (n - 1)d).$$

Example *The arithmetic sum*

$$1 + 3 + 5 + 7 + 9 + \ldots + 501$$

has 251 *terms, because*

$$u_n = 501 \textit{ and } u_n = 1 + 2(n - 1),$$

yielding $n = 251$.

The sum is thus $\frac{n}{2} \times$ (*first term* $+$ *last term*)
$= 251/2 \times (1 + 501) = 63001$.

Geometric sums

The *geometric sum* with initial term a, common ratio r and n terms is the sum of the first n terms of a geometric sequence, that is

$$a + ar + ar^2 + \ldots + ar^{n-1}.$$

Provided $r \neq 1$, this geometric sum is

$$a \times \frac{r^n - 1}{r - 1}.$$

(If $r = 1$ the sum is just the arithmetic sum with n terms, initial term a and common difference 0, and thus has sum na.)

Example *The geometric sum*

$$1 + 2 + 4 + \ldots 2048$$

has 12 *terms, because*

$$u_n = 2048 \textit{ and } u_n = 1 \times 2^{n-1},$$

yielding $n = 12$.

The sum is thus $1 \times \frac{2^{12}-1}{2-1} = 4095$.

Summation notation

We often have to write the sum of a sequence of numbers. An example of this is the sum of the first 100 integers which can be written as:

$$1 + 2 + 3 + \ldots + 100$$

or alternatively as:

$$\sum_{r=1}^{100} r$$

In general we can represent the sum of the first n terms of a sequence u_r by:

$$\sum_{r=1}^{n} u_r$$

Example *Summation notation:*

$$\sum_{r=1}^{4} (3r - 1) = 2 + 5 + 8 + 11$$

This is an example of the sum of the sequence defined by $u_1 = 2$ and $u_{r+1} = u_r + 3$ from the first term to the 4th term. The values 1 and 4 are called the limits of the summation.

An arithmetic and a geometric sum expressed in Σ-notation

$$\sum_{r=0}^{n} r = \frac{(n+1)n}{2}$$

$$\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r - 1}$$

A useful sum expressed in Σ-notation

The binomial coefficient $\binom{n}{r}$ is defined as

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

These can be used to evaluate expressions $(x+y)^n$:

The formula is known as the **Binomial Theorem**:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Example

$\binom{5}{0} = 1$ $\quad \binom{5}{1} = 5$ $\quad \binom{5}{2} = \frac{5!}{3!\,2!} = \frac{5\cdot 4}{2\cdot 1} = 10$ $\quad \binom{5}{3} = \binom{5}{2} = 10$ $\quad \binom{5}{4} = \binom{5}{1} = 5$ $\quad \binom{5}{5} = 1$

so

$$(x+y)^5 = \sum_{r=0}^{5} \binom{5}{r} x^{5-r} y^r$$

$$= \binom{5}{0} x^5 y^0 + \binom{5}{1} x^4 y^1 + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} x^1 y^4 + \binom{5}{5} x^0 y^5$$

$$= x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$$

Working with sums in Σ-notation

Finite sums can be manipulated in Σ-notation in exactly the same way as we do in 'long' notation:

Formula 1: $$\sum_{r=n}^{m}(a_r+b_r) = \sum_{r=n}^{m}a_r + \sum_{r=n}^{m}b_r$$

Formula 2: $$\sum_{r=n}^{m}c\,a_r = c\sum_{r=n}^{m}a_r$$

Example

$$\sum_{r=2}^{4}(a_r+b_r) = (a_2+b_2)+(a_3+b_3)+(a_4+b_4)$$

$$= (a_2+a_3+a_4)+(b_2+b_3+b_4)$$

$$= \sum_{r=2}^{4}a_r + \sum_{r=2}^{4}b_r$$

Example

$$\sum_{r=2}^{5}3a_r = 3a_2+3a_3+3a_4+3a_5$$

$$= 3(a_2+a_3+a_4+a_5)$$

$$= 3\sum_{r=2}^{5}a_r$$

Example *The arithmetic sum*

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

This sum can be used to compute many other sums like for example

$$\begin{aligned}
\sum_{r=1}^{1000}(2r+1) &= \sum_{r=1}^{1000}(2r) + \sum_{r=1}^{1000} 1 \\
&= 2\sum_{r=1}^{1000} r + \sum_{r=1}^{1000} 1 \\
&= \frac{2\cdot 1000\cdot 1001}{2} + 1000 \\
&= 1000\cdot 1001 + 1000 \\
&= 1001000 + 1000 \\
&= 1002000.
\end{aligned}$$

Example *The arithmetic sum*

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

can also be used to compute arithmetic sums that do not start from 0 like for example

$$\begin{aligned}
\sum_{r=21}^{1000}(2r+1) &= \sum_{r=1}^{1000}(2r+1) - \sum_{r=1}^{20}(2r+1) \\
&= 1002000 - 2\sum_{r=1}^{20} r - \sum_{r=1}^{20} 1 \\
&= 1002000 - 20 \cdot 21 - 20 \\
&= 1002000 - 420 - 20 \\
&= 1002000 - 440 \\
&= 1001560.
\end{aligned}$$

Changing variable in a sum (sv: variabelbyte)

It is often possible to write the same sum in a number of different ways.

Example *Changing variables in a sum:*

$$\sum_{r=1}^{8}(3r+2) = 5+8+11+14+17+20+23+26$$

$$\sum_{s=2}^{9}(3s-1) = 5+8+11+14+17+20+23+26$$

To prove that these are equivalent we use the change of variable given by $s = r+1$. ($\Leftrightarrow r = s-1$)
First change the limits of the sum:

$$r = 1 \Rightarrow s = 2$$

$$r = 8 \Rightarrow s = 9$$

Then change the 'body' of the sum:

$$3r+2 = 3(s-1)+2 = 3s-1$$

Hence

$$\sum_{r=1}^{8}(3r+2) = \sum_{s=2}^{9}(3s-1)$$