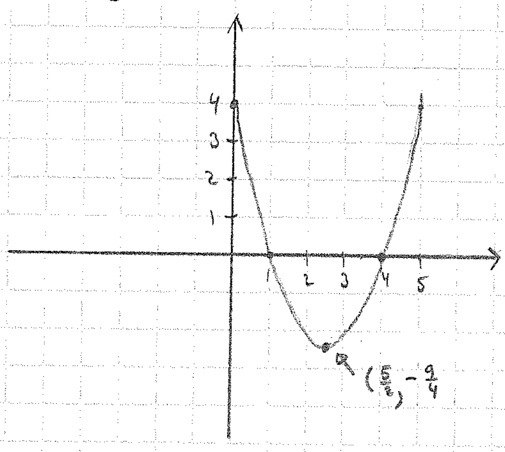
MODELLTENTA

Exercise 1

(a) $x^2-5x+4 = (x-\frac{5}{2})^2-(\frac{5}{2})^2+4 = (x-\frac{5}{2})^2-\frac{25}{4}+\frac{16}{4} = (x-\frac{5}{2})^2-\frac{9}{4}.$

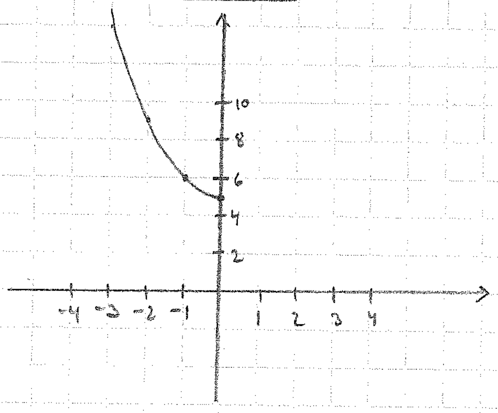
(b) (i) The graph of f is:



- $V_f = [-\frac{9}{4}, 4]$
- f is not invertible because f is not 1-1,

  e.g. $f(0)=f(5)=4$ while $0 \neq 5$.

(ii) $\underline{V_g = [5, \infty[}$



g is invertible because g is 1-1

as it is strictly decreasing

if $x_1 > x_2$ then $x_1^2+5 > x_2^2+5$

$y = x^2+5$
$\Downarrow$
$y-5 = x^2$
$\Downarrow$
$x = \pm\sqrt{y-5}$
$\Downarrow$
$x = -\sqrt{y-5}$ as $x \leq 0$

So $g^{-1}: [5,\infty[\to]-\infty, 0]$

$\underline{g^{-1}(x) = -\sqrt{x-5}}$

$\underline{D_{g^{-1}} = V_g = [5,\infty)}$

$\underline{V_{g^{-1}} = D_g = (-\infty, 0]}$

Exercise 2

(a) $\lim\limits_{x\to\infty} \frac{3x-1}{x-2} = \lim\limits_{x\to\infty} \frac{3-\frac{1}{x}}{1-\frac{2}{x}} = \frac{3-0}{1-0} = 3$

$\lim\limits_{x\to\infty} \frac{(3x-1)^3}{(x-3)^5} = 0$ as the numerator has degree 3 < degree 5 of denominator.

(b) $-1 \leq \cos\left(\frac{1}{x-2}\right) \leq 1$, defined for all $x \neq 2$.

For all $x \neq 2$ $(x-2)^2 > 0$, so for $x \neq 2$ we have

$-(x-2)^2 \leq (x-2)^2 \cos\left(\frac{1}{x-2}\right) \leq (x-2)^2$

Now $\lim\limits_{x\to 2} -(x-2)^2 = 0$ and $\lim\limits_{x\to 2} (x-2)^2 = 0$

so by the Sandwich Theorem $\lim\limits_{x\to 2} (x-2)^2 \cos\frac{1}{x-2} = 0$ also

Exercise 3

(a) f is continuous at $x = x_0$ if • $\lim\limits_{x\to x_0} f(x)$ exists and • $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

(b) By (a) $x_0 \in D_f$ if f is continuous.

However, the function $f(x) = \frac{x^2-25}{x-5}$ is not defined at $x=5$,

hence it is not continuous at $x=5$.

(c) $\lim\limits_{x\to 5+} g(x) = \lim\limits_{x\to 5+} b/x - 2 = \frac{b}{5} - 2$

$\lim\limits_{x\to 5-} g(x) = \lim\limits_{x\to 5-} (3-x^2) = 3-5^2 = -22$

for g to be continuous at $x=5$ we must have

$\lim\limits_{x\to 5+} g(x) = \lim\limits_{x\to 5-} = g(5)$ $a = g(5) = \frac{b}{5} - 2 = -22$ and thus $a = -22$, $b = -100$

Exercise 4

(a) $b = \frac{16-(-2)}{2} = 9$

$a = 16 - b = 16 - 9 = 7$

So $\{x \in \mathbb{R} : |x-7| \leq 9\} = [-2, 16]$

(b)

$x > 2$	$x < 2$
$(x+3)(x-2) \leq 2x$	$(x+3)(x-2) \geq 2x$
$x^2+x-6 \leq 2x$	$x^2+x-6 \geq 2x$
$x^2-x-6 \leq 0$	$x^2-x-6 \geq 0$
$(x-3)(x+2) \leq 0$	$(x-3)(x+2) \geq 0$
$-2 \leq x \leq 3$	$x \leq -2$ or $x \geq 3$
part sol:	part sol:
$2 < x \leq 3$	$x \leq -2$

So solution is $x \in \,]2,3] \cup]-\infty,-2]$

(c) $\frac{5x-11}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4} = \frac{A(x-4)+B(x-1)}{(x-1)(x-4)} = \frac{(A+B)x-(4A+B)}{(x-1)(x-4)}$

By comparing numerators we get $5x-11 = (A+B)x-(4A+B)$ so

$$\begin{bmatrix} A+B = 5 \\ 4A+B = 11 \end{bmatrix}$$

subtract the two equations

$$\Downarrow$$

$$-3A = -6$$

$$A = 2$$

$$\Downarrow$$

$$2+B = 5$$

$$\Downarrow$$

$$B = 3$$

So $\frac{5x-11}{(x-1)(x-4)} = \frac{2}{x-1} + \frac{3}{x-4}$

Exercise 5

(a) $$S = \sum_{r=17}^{1004}(2r+1) = 2\sum_{r=17}^{1004} r + \sum_{r=17}^{1004} 1 = 2\sum_{r=1}^{1004} r - 2\sum_{r=1}^{16} r + (1004-16)$$

$$= \frac{\cancel{2}\cdot 1004\cdot 1005}{\cancel{2}} - \frac{\cancel{2}\cdot 16\cdot 17}{\cancel{2}} + 988 = \underline{\underline{1009736}}$$

(b) $$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=10}^{100}(i-9)^2 = \sum_{r=1}^{91} r^2 = \frac{91\cdot 92\cdot (2\cdot 91+1)}{6} = \underline{\underline{255346}}$$

$r = i-9$

$i=10 \Rightarrow r=1$

$i=100 \Rightarrow r=91$

Exercise 6

$$f'(x) = \frac{d}{dx}\left(4x^3 + 2\ln(5x)\right) = 12x^2 + 2\cdot\frac{1}{\cancel{5}x}\cdot\cancel{5} = \underline{\underline{12x^2 + \frac{2}{x}}}$$

$$g'(x) = \frac{d}{dx}\left(e^{3x^2}\cos x\right) = e^{3x^2}(-\sin x) + 6x e^{3x^2}\cos x = \underline{\underline{e^{3x^2}(6x\cos x - \sin x)}}$$

$$h'(x) = \frac{\frac{1}{x}\cdot\sin x - (\ln x)(\cos x)}{\sin^2 x} = \underline{\underline{\frac{1}{x\sin x} - \frac{\ln(x)\cdot\cos(x)}{\sin^2 x}}}$$

Exercise 7

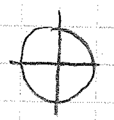
(a) $\sin(4x) - 2\sin(2x) = 2\cos 2x \cdot \sin 2x - 2\sin 2x = 2\sin 2x\,(\cos 2x - 1)$

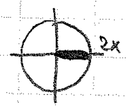
so the equation is $2\sin 2x\,(\cos 2x - 1) = 0$

$\Downarrow$

$2\sin 2x = 0$ or $\cos 2x = 1$

$\Downarrow$

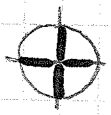


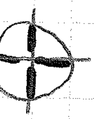


$2x = k\pi,\ k \in \mathbb{Z}$ or $2x = k \cdot 2\pi\ \ k \in \mathbb{Z}$

$\Downarrow$

$x = k\frac{\pi}{2}$ $\qquad$ $x = k\pi$





combining these gives $\underline{\underline{x = k\frac{\pi}{2}\ \ k \in \mathbb{Z}}}$

(b) $\sqrt{x-9} - \sqrt{x} + 1 = 0$

$\Updownarrow$

$\sqrt{x-9} = \sqrt{x} - 1$

$\Downarrow$

$x - 9 = x - 2\sqrt{x} + 1$

$\Downarrow$

$2\sqrt{x} = 10$

$\Downarrow$

$\sqrt{x} = 5$

$\Downarrow$

$x = 25$

check for fake root: $\sqrt{25-9} - \sqrt{25} + 1 = 4 - 5 + 1 = 0$ ✓

So solution is $x = 25$

(c) $\ln(x^2-1) - \ln(x-1) = 0$

$\Downarrow$

$\ln \frac{x^2-1}{x-1} = 0$

$\Downarrow$

$\ln(x+1) = 0$

$\Downarrow$

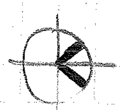
$x = 0$

$\Downarrow$

no solutions as $x=0$ is a fake root

(d) $2\cos 3X \quad \sqrt{3}$

$\Downarrow$



$\cos 3x = \sqrt{3}/2$

$\Downarrow$

$3x = \pm\frac{\pi}{6} + k \cdot 2\pi$

$\Downarrow$

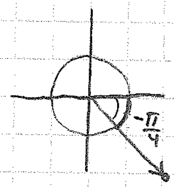
$\underline{\underline{x = \pm\frac{\pi}{18} + k\frac{2\pi}{3}}}$

Exercise 8

(a) $(3-2i)(4+3i) = 12-8i+9i-6i^2 = \underline{18+i}$

$$\frac{1+5i}{3-2i} = \frac{(1+5i)(3+2i)}{(3+2i)(3-2i)} = \frac{3+10i^2+15i+2i}{13} = \underline{\underline{\frac{-7}{13} + \frac{17}{13}i}}$$

(b) $z = 1-i$



$\arg z = -\frac{\pi}{4}$

$|z| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$z = \sqrt{2}\, e^{-\frac{\pi}{4}i}$

$\frac{-10\pi}{4} = \frac{-8\pi}{4} - \frac{2\pi}{4} = -2\pi - \frac{\pi}{2}$.

$$z^{10} = (\sqrt{2})^{10} e^{-\frac{10\pi}{4}i} = 32e^{-\frac{\pi}{2}i} = \underline{-32i}$$ by De Moivre's Thm.

(c) • $w^5 = 32i$ $\qquad \arg w^5 = \arg 32i = \frac{\pi}{2}$

$|w^5| = 32|i| = 32$

so $w^5 = 32e^{i\frac{\pi}{2}}$

one solution is $w = 2e^{i\frac{\pi}{10}}$, and since the roots are evenly distributed around a circle, the others occur at angles $\frac{\pi}{10}+\frac{2\pi}{5}$, $\frac{\pi}{10}+\frac{4\pi}{5}$, $\frac{\pi}{10}+\frac{6\pi}{5}$, $\frac{\pi}{10}+\frac{8\pi}{5}$

$w = \underline{\underline{2e^{i\frac{\pi}{10}}}}, \underline{\underline{2e^{i\frac{\pi}{2}}}}, \underline{\underline{2e^{i\frac{9\pi}{10}}}}, \underline{\underline{2e^{i\frac{13\pi}{10}}}}, \underline{\underline{2e^{i\frac{17\pi}{10}}}}$

• $z^3-3z^2+4z-2=0$ $\qquad$ By the rational root test, the only possible rational roots here are $\pm1, \pm2$.

1 turns out to be a root as $1^3-3\cdot1^2+4\cdot1-2=0$

Divide by $z-1$:

$$\begin{array}{r|l} & z^2-2z+2 \\ \hline z-1 & z^3-3z^2+4z-2 \\ & \underline{z^3-z^2} \\ & -2z^2+4z-2 \\ & \underline{-2z^2+2z} \\ & 2z-2 \\ & \underline{2z-2} \\ & 0 \end{array}$$

Solve $z^2-2z+2=0$

$(z-1)^2 = -1$

$\Downarrow$

$z-1 = \pm i$

$\Downarrow$

$z = 1\pm i$

So the three solutions are $\underline{\underline{z=1,\ z=1+i,\ z=1-i}}$

Exercise 9

$x^2-3x+2=0$

$\Updownarrow$

$(x-2)(x-1)=0$

$\Updownarrow$

$x=1$ or $x=2$

(a) So $D_f = \underline{\mathbb{R}\setminus\{1,2\}} = \{x\in\mathbb{R} \mid x\neq 1, x\neq 2\}$

(b) $f(x)=0 \Rightarrow$ numerator $=0 \Rightarrow x^4-7x^3+17x^2-17x+6 = p(x) = 0$

By RRT, possible roots are $\pm1, \pm2, \pm3, \pm6$

$p(1) = 1-7+17-17+6 = 0$ so $x=1$ is a root

$p(-1) = 1+7+17+17+6 \neq 0$

$p(2) = 16-56+68-34+6 = 0$ so $x=2$ is a root

$p(-2) = 16+15+76+34+6 \neq 0$

$p(3) = 81-7\cdot27+17\cdot9-17\cdot3+6 = 0$ so $x=3$ is a root

So by the Factor Theorem $(x-1)(x-2)(x-3)$ divides $p(x)$

$(x-1)(x-2)(x-3) = (x^2-3x+2)(x-3) = (x^3-3x^2+2x-3x^2+9x-6) = x^3-6x^2+11x-6$

$$\begin{array}{r|l} & x-1 \\ \hline x^3-6x^2+11x-6 & x^4-7x^3+17x^2-17x+6 \\ & \underline{x^4-6x^3+11x^2-6x} \\ & -x^3+6x^2-11x+6 \\ & \underline{-x^3+6x^2-11x+6} \\ & 0 \end{array}$$

So $p(x) = (x-1)(x-1)(x-2)(x-3)$ and thus $f(x) = \dfrac{(x-1)(x-1)(x-2)(x-3)}{(x-2)(x-1)}$

So $f(x)=0 \Rightarrow \underline{x=3}$ as $f(x)$ is not defined for $x=1, x=2$.